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A single-resource revenue management problem with random resource consumptions

W Zhuang[1], M Gumus[2*] and D Zhang[3]

[1]*School of Management, Xiamen University, Fujian, China;* [2]*Desautels Faculty of Management, McGill University, Montreal, Quebec, Canada; and* [3]*Leeds School of Business, University of Colorado at Boulder, Boulder, CO, USA*

We study a single-resource multi-class revenue management problem where the resource consumption for each class is random and only revealed at departure. The model is motivated by cargo revenue management problems in the airline and other shipping industries. We study how random resource consumption distribution affects the optimal expected profit and identify a preference acceptance order on classes. For a special case where the resource consumption for each class follows the same distribution, we fully characterize the optimal control policy. We then propose two easily computable heuristics: (i) a class-independent heuristic through parameter scaling, and (ii) a decomposition heuristic that decomposes the dynamic programming formulation into a collection of one-dimensional problems. We conduct extensive numerical experiments to investigate the performance of the two heuristics and compared them with several widely studied heuristic policies. Our results show that both heuristics work very well, with class-independent heuristic slightly better between the two. In particular, they consistently outperform heuristics that ignore demand and/or resource consumption uncertainty. Our results demonstrate the importance of considering random resource consumption as another problem dimension in revenue management applications.

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1. Introduction

The majority of revenue management literature focuses on selling scarce capacity units to customers that have *deterministic* resource consumption requirements but may potentially have different revenue contributions. More specifically, the number of units of capacity an accepted customer will consume is generally assumed to be known. This assumption is realistic for many revenue management applications, where customer resource consumption requirements are quite similar and can be easily defined in terms of standard units, such as the number of seats, rooms, cars, etc. However, it is not natural in cases where either the standardization of this requirement is difficult or the requirement itself is not predictable. In this paper, our goal is to study the impact of the uncertainty in resource consumption requirements on revenue management decisions.

Uncertainty in resource consumption plays a significant role for many revenue management applications. For example, in the air-cargo business,[1] carriers often do not know physical attributes (such as weight, and volume) of the cargo to be shipped at the time of booking decision. This might be due to the fact that a carrier sells her capacity through multiple channels that differ in terms of both their contractual agreements with the carrier and cargo characteristics. In a typical air-cargo business, carriers can receive the cargo either from intermediaries (so called *forwarders*) or directly from shippers. Usually, forwarders have long-term relationships with the carrier and generate cargo with relatively predictable characteristics, whereas shippers interact with the carrier on an *ad hoc* basis and can demonstrate significant variations in terms of cargo specifications; see Gupta (2008), and references therein for additional details on air-cargo business.

Similarly, hotels and passenger airlines sell their capacities to customers either directly (using online or off-line channels) or through intermediaries (so-called *travel agencies*). In these industries, resource consumption

Correspondence: M Gumus, Operations Management Area, Desautels Faculty of Management, McGill University, 1001 Sherbrooke St. West, Montreal, Quebec, Canada H3A 1G5.

[1]According to a recent forecast report (Boeing, 2009), the world air cargo business, a 40-billion dollar industry worldwide, has been constantly growing at around 5% per year and is expected to triple in the next 20 years.

uncertainty comes into the picture in the form of *group booking*, a general practice commonly used by travel agencies. More specifically, when a travel agency makes a group reservation, it does not usually specify the exact number of seats or rooms *a priori*. In fact, a typical group booking contract for a conference hotel (Lakehead, 2009) involves an estimate for the number of rooms to be reserved at the time of booking decision and the penalty/cancellation costs if the realized numbers do not match the estimates.

All of these cases underline the need for effective strategies to deal with resource consumption uncertainty. Indeed, inability to predict the very final resource consumption prior to a booking decision may hamper a decision maker's ability to extract maximum achievable revenues from available capacities. In addition, rising commodity prices and distress in the financial sector significantly increase operating costs in all industries, which, in turn, implies that letting even one unit of capacity sit idle could adversely affect a company that is already at break-even.

This paper has two objectives. Our first goal is to analyse how the variability in resource consumption impacts both optimal booking decisions and profits. Second, we propose two heuristics that specifically use the information regarding the degree of randomness in resource consumption to optimize booking decisions and compare them with ones that ignore this randomness. In order to achieve these objectives, we develop a dynamic programming model with random resource consumptions. To specifically capture the differences among various arrival streams (eg, selling directly or through intermediaries as in air-cargo example), we assume that there are multiple customer classes, each of which has different revenue and resource consumption characteristics.

Using this model, we show that total expected profit decreases in the volatility of resource consumption distribution. However, observing that a general multi-class version of the problem suffers from the curse of dimensionality, we restrict our attention to various special cases. First, we characterize the structure of the optimal booking policy for a two-class problem and show that it involves state-dependent thresholds (ie, *booking limits)* for each customer class. Contrary to the intuition, we show that the booking limits for different classes are not necessarily monotone in the number of reservations belonging to that class. Optimality of the booking limit policy is also quite sensitive to the number of customer classes. Through a counter-example, we show that it is no longer optimal when we consider a three-class case. This shows that the characterization of the optimal policy is not only computationally, but also analytically, very difficult.

At this stage, aiming to implement our second objective above, we change our focus and develop two heuristics that capture the essence of randomness in the resource consumption requirements in a tractable fashion. In our first heuristics, we propose a *parameter-scaling scheme* that reduces the multi-class problem into a single-class problem. This is implemented by generating a common distribution from the first- and second-moment of the random resource consumption distribution of each class. We show that the resulting heuristics is optimal when all uncertainties in resource consumptions can be represented by a common distribution function. In the second heuristic, we take a different approach; instead of combining resource consumptions, we *decompose* them into independent single-class problems that are easily computable. Through extensive numerical experiments, we show that the two heuristics can significantly outperform several widely used approaches that ignore the randomness in resource consumption. The parameter-scaling and decomposition approaches have been frequently employed in revenue management literature to break the curse of dimensionality in dynamic resource allocation problems. However, to the best of our knowledge, they have not been used in single-resource revenue management with random resource consumptions, which defines one of the contributions of this paper.

The remainder of the paper is structured as follows. In Section 2, we review the related literature. In Section 3, we develop our model framework and in Section 4, we analyse its structural properties. In Section 5, we propose our heuristics and in Section 6, we test their performances in an extensive numerical study. Finally, in Section 7, we conclude.

2. Literature review

Our work is closely related to three streams of literature on cargo revenue management, stochastic knapsack, and overbooking problems. Below, we review each stream in detail and discuss their connections to our work.

The models in cargo revenue management literature take the ones developed for passenger revenue management as a starting point and add the following cargo-specific extensions: (a) uncertain capacity, (b) multi-dimensional capacity, (c) random resource consumptions, (d) itinerary control, and (e) allotments for forwarders *versus* free-sale (Kasilingam, 1997; Billings *et al*, 2003; Slager and Kapteijns, 2004; Amaruchkul *et al*, 2007; Becker and Dill, 2007, etc). Since (d) and (e) are related to network revenue management and free-sale capacity management, respectively, both of which are quite different from our focus, we will limit our attention to the papers in (a)-(c). One of the closest model to our work in these categories is that of Amaruchkul *et al* (2007). They formulate a two-dimensional Markov Decision Process (MDP) for the single-leg air-cargo revenue management problem with random weights and volumes and develop a heuristic that decomposes the two-dimensional MDP into two single-dimensional MDPs. For the same problem, Huang and Chang (2010) develop a heuristic that estimates the expected revenue by jointly sampling weight and volume and show that it performs around 1% better than the decomposition heuristics of Amaruchkul *et al* (2007). Taking a different approach, Han *et al* (2010) propose a bid-price control policy for the same problem, and show that it outperforms both static bid-price control and first-come first-serve control policies. Xiao and Yang (2010) propose a continuous-time MDP for the revenue management problem with two-dimensional capacity and characterize that the optimal control policy is of threshold-type under some concavity assumptions. Two features distinguish this work from the above papers. First, we concentrate on the impact of random resource consumptions on decisions and profits, whereas the above models mostly focus on multi-dimensionality issues with regards to the capacity and demand. Second, the heuristics in this paper are specifically designed to use the information regarding the degree of uncertainty in resource consumptions, which is generally ignored by the existing heuristics in this stream.

The second stream of related literature is the one developed for stochastic knapsack problems. Papastavrou *et al* (1996) study a stochastic knapsack problem with deadlines, where both weights and rewards are random. They show that the optimal control policy is of threshold-type. Kleywegt and Papastavrou (1998), and Kleywegt and Papastavrou (2001) extend the analysis in Papastavrou *et al* (1996) to the infinite horizon case with deterministic and random capacity requirements, respectively. They investigate the monotonicity and convexity properties of the problem and prove the optimality of threshold-type control policy. Van Slyke and Young (2000) develop a continuous-time MDP formulation for finite-horizon stochastic knapsack, and characterize the optimal return function and acceptance strategy. There are also two features that distinguish our work from the literature on stochastic knapsack problems. In this stream, it is generally assumed that the resource consumptions for multiple classes are distributed identically, whereas our model allows for heterogeneity among the resource consumptions. Second, overbooking is not allowed in this literature, whereas it is implicitly captured in this paper through a terminal penalty function.

The last stream of literature related to our paper examines overbooking, mostly in the contexts of passenger and cargo revenue management. Subramanian *et al* (1999) study a dynamic model of airline yield management with overbooking, cancelations and no-shows. They show that the state-and time-dependent booking limit policy is optimal. Moussawi and Cakanyildirim (2006) formulate a two-dimensional cargo overbooking problem and provide aggregate formulation as an effective approximation, which is simple and yields the same optimal overbooking curve. Luo *et al* (2009) study single-period air-cargo overbooking problem to obtain simple solutions. The overbooking literature generally assumes unit resource consumption for each customer and the uncertainty in resource consumption results from customer cancelation, and therefore is of an all-or-nothing type. Our work can be viewed as a generalization of this research literature where we allow (i) multi-unit resource consumption, and (ii) more general continuously distributed resource consumption uncertainty.

3. Model formulation

Consider a single resource with capacity c. Demand is divided into n customer classes. The booking horizon is divided into T discrete time periods. Time runs forward, so the first time period is period 1 and the last period is period T. Period $T+1$ is used to represent the end of the horizon. In each time period t, there is a class i customer arrival with probability λ_i. The probability of no customer arrival is $\lambda_0 = 1-\sum_{i=1}^{n}\lambda_i$.[2] Note that we assume implicitly that there is, at most, one customer arrival in each period. The resource consumption of a class i arrival is random and denoted by A_i with mean μ_i and standard deviation σ_i. Let $F_i(\cdot)$ be the cumulative distribution function for A_i We assume that all A_i's are independent. The resource consumption A_i of a class i arrival, if accepted, is realized at time $T+1$. Let $\tilde{r}_i$ be the unit revenue for class-i. Then, the total expected contribution from a class i customer, denoted by r_i, can be written as $r_i = \tilde{r}_i\mu_i$. Without loss of generality, we assume that the expected contributions are in descending order, that is $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_n$. Let y be the total resource requirement of all accepted customers realized at time $T+1$. If $y > c$, a unit penalty cost of b is charged for each unit exceeding capacity, that is, the penalty cost paid at period $T+1$ is equal to $h(y)$, where

$$h(y) = b(y-c)^{+}. \tag{1}$$

Note that all our results hold as long as $h(\cdot)$ is convex and non-decreasing. Our objective is to find an optimal control policy to maximize total expected revenue minus the expected penalty cost. We consider a dynamic programming formulation for the problem. The state can be denoted by an n-dimensional *reservation* vector $\mathbf{x} = (x_1, \ldots, x_n)$, where x_i is the total number of reservations accepted for class i. Since there can be at most one arrival in each period, we have $x_i \leqslant t$ for all i in period t. Upon a class i arrival in period t, we decide whether to accept or reject the customer. Let $u_i = 1$ or $u_i = 0$ represent the action of accepting or rejecting an arriving class-i

[2]We assume the arrival probability for each customer class does not depend on t only for notational simplicity; all our results holds for non-stationary arrival probabilities.

request, respectively. Therefore, the dynamic program has finite state space and finite action space.

Let $v_t(\mathbf{x})$ be the maximum total expected profit from period t onwards given state $\mathbf{x}$. First, the terminal value in period $T+1$ reflects the expected terminal penalty given by

$$v_{T+1}(\mathbf{x}) = -\mathbf{E}[h(Y_{\mathbf{x}})] \qquad \forall \mathbf{x}, \tag{2}$$

with $Y_{\mathbf{x}} = \sum_{i=1}^{n} \sum_{k=1}^{x_i} A_{ki}$, where A_{ki} denotes the resource requirement of the kth customer in class-i. Then, for each period $t \leqslant T$ and state $\mathbf{x}$, the dynamic programming optimality equation can be written as

$$\begin{aligned} v_t(\mathbf{x}) &= \sum_{i=1}^{n} \lambda_i \max\{v_{t+1}(\mathbf{x}+\mathbf{e}_i) + r_i,\ v_{t+1}(\mathbf{x})\} + \lambda_0 v_{t+1}(\mathbf{x}) \\ &= \max_{\mathbf{u}\in\{0,1\}^n} \sum_{i=1}^{n} \lambda_i u_i [r_i - \Delta_i v_{t+1}(\mathbf{x})] + v_{t+1}(\mathbf{x}), \end{aligned} \tag{3}$$

where $\Delta_i v_{t+1}(\mathbf{x}) = v_{t+1}(\mathbf{x}) - v_{t+1}(\mathbf{x}+\mathbf{e}_i)$ is the opportunity cost of accepting a class-i request in period t.

The dynamic programming formulation introduced in this section has several interesting special cases. When $n=1$, the model can be viewed as a variant of the widely studied newsvendor model with finite initial inventory. Note that when there is a single customer class, customer requests will be accepted until a booking limit is reached. Interestingly, this booking limit can be different from the capacity level c. In the extreme case where $b=0$, all customer requests will be accepted. Using revenue management terminology, this means that overbooking is implicitly modelled in the dynamic programming formulation. Indeed, since demand resource requirements are uncertain, the boundary between booking control and overbooking is blurred. In this sense, our dynamic programming formulation above implicitly models both booking control and overbooking.

Another interesting case is given when $A_i = 1$ with probability 1 for each class i and terminal penalty function, that is, $h(y)$, is defined as follows:

$$h(y) = \begin{cases} 0 & \text{if } y \leqslant c, \\ +\infty & \text{if } y > c. \end{cases}$$

In this case, no overbooking will be allowed. The model reduces to the widely studied single-leg revenue management problem (Lee and Hersh, 1993). In the Lee and Hersh (1993) model, the only uncertainty comes from the demand arrival process. The model introduced in this section adds another uncertainty dimension—the uncertainty regarding resource requirements. One interesting research question, which we will study in this paper, is whether or not the added dimension plays an important role in terms of revenue and policy performance.

It is straightforward to show that in (3), an optimal policy is to accept class i at time t whenever $r_i \geqslant \Delta_i v_{t+1}(\mathbf{x})$. However, this observation is not sufficient for the efficient solution of the model due to the curse of dimensionality. In order to gain intuition that will enable us to devise efficient solution approaches for the problem, we explore the structural properties of the model in the following section.

4. Structural properties

In this section, we explore structural properties of the problem. We start with the general model with n customer classes in Section 4.1. We then restrict our attention to two special cases, the two-class model with class-dependent (ie, heterogenous) resource consumptions in Section 4.2 and the n-class model with class-independent (ie, homogenous) resource consumptions in Section 4.3, respectively.

4.1. General model: n-class with class-dependent resource consumptions

Define the control operators

$$T_k v_t(\mathbf{x}) = \max\{v_t(\mathbf{x}+\mathbf{e}_k) + r_k,\ v_t(\mathbf{x})\}, \tag{4}$$

$$v_t(\mathbf{x}) = \sum_{k=1}^{n} \lambda_k T_k v_{t+1}(\mathbf{x}) + \lambda_0 v_{t+1}(\mathbf{x}). \tag{5}$$

We will use some stochastic comparison concepts. To this end, we introduce the following definition and lemma (see Shaked and Shanthikumar (1994) for the following definition and proof of Lemma 1).

Definition 1 Let X and Y be two random variables.

(a) X is said to be larger than Y in the *stochastic order*, that is, $X \geqslant_{st} Y$, if and only if $\mathrm{E}[\phi(X)] \geqslant \mathrm{E}[\phi(Y)]$ for all non-decreasing real-valued functions ϕ;
(b) X is said to be larger than Y in the *convex order*, that is, $X \geqslant_{cx} Y$, if and only if $\mathrm{E}[\phi(X)] \geqslant \mathrm{E}[\phi(Y)]$ for all convex real-valued functions ϕ.

Lemma 1 *Stochastic order and convex order have the following properties*:

(a) *If* $X \geqslant_{st} Y$, *then* $\mathbf{E}[X] \geqslant \mathbf{E}[Y]$;
(b) *If* $X \geqslant_{cx} Y$, *then* $\mathbf{E}[X] = \mathbf{E}[Y]$ *and* $Var[X] \geqslant Var[Y]$;
(c) *Let* $X_1, \ldots, X_n$ *be a set of independent random variables and* $Y_1, \ldots, Y_n$ *be another set of independent random variables. If* $X_i \geqslant_{st} Y_i$ *for* $i=1,\ldots,n$, *then* $\sum_{i=1}^{n} X_i \geqslant_{st} \sum_{i=1}^{n} Y_i$. *If* $X_i \geqslant_{cx} Y_i$ *for* $i=1,\ldots,n$, *then* $\sum_{i=1}^{n} X_i \geqslant_{cx} \sum_{i=1}^{n} Y_i$.

Parts (a) and (b) of Lemma 1 say that stochastic order means stochastically larger while convex order means stochastically more variable. Part (c) implies that both stochastic order and convex order are closed under convolutions.

Theorem 1 *For any non-decreasing convex penalty function $h(\cdot)$, we have*

(a) *If $Y_{\tilde{\mathbf{x}}} \geqslant_{cx} Y_{\mathbf{x}}$ and/or $Y_{\tilde{\mathbf{x}}} \geqslant_{st} Y_{\mathbf{x}}$, then $v_t(\tilde{\mathbf{x}}) \leqslant v_t(\mathbf{x})$;*
(b) *If $A_i \geqslant_{cx} A_j$ and/or $A_i \geqslant_{st} A_j$, then $v_t(\mathbf{x}+\mathbf{e}_i) \leqslant v_t(\mathbf{x}+\mathbf{e}_j)$ and $\Delta_i v_t(\mathbf{x}) \geqslant \Delta_j v_t(\mathbf{x})$;*
(c) *If $A_1 \leqslant_{cx} \cdots \leqslant_{cx} A_n$ and/or $A_1 \leqslant_{st} \cdots \leqslant_{st} A_n$, it is optimal to accept classes $1, 2, \ldots, i-1$ whenever it is optimal to accept, accept class-i.*

Theorem 1 characterizes the variance effect and mean effect of demand. Part (a) implies that the optimal expected profit-to-go is lower when the total resource consumption of accepted reservations is larger in the convex and usual stochastic order. Part (b) says that if the resource consumption of class-*i* is bigger or more variable than that of class-*j*, then the opportunity cost of accepting a class-*i* request is larger than that of a class-*j*. Part (c) shows that under certain order of random resource consumptions and expected contributions, customer classes can be ordered in the sense that if it is optimal to accept the less profitable class in a certain state, then an arriving request of the more profitable class should also be accepted.

Unfortunately, a simple characterization of the optimal booking policy under general *n*-class model is extremely complicated. Recall from the optimality equation 3, an optimal policy is to accept class *i* at time *t* whenever the revenue contribution from class *i* exceeds the opportunity cost incurred by accepting it, that is, $r_i \geqslant \Delta_i v_{t+1}(\mathbf{x})$. The commonly used approach for revenue management models in the literature is to explore whether the opportunity cost function is monotone. If it is shown to be monotone, then it implies existence of a simple and implementable policy, known as *booking limit* policy. Namely, in this policy, one can define a threshold (called booking limit) for each class *i* such that it is optimal to accept an arrival as long as the total number of accepted reservation does not exceed the threshold. In the following example, we show that for a general *n*-class problem, where $n \geqslant 3$, such a simple booking limit policy does not exist.

Example 1 Consider a problem with three customer classes. The random resource consumption requirements A_1, A_2 and A_3, are independent and normally distributed with $(\mu_1, \sigma_1) = (20, 6)$, $(\mu_2, \sigma_2) = (10, 2)$ and $(\mu_3, \sigma_3) = (3, 1)$, respectively. Assume that $\lambda_1 = 0.3$, $\lambda_2 = 0.5$, $\lambda_3 = 0.1$, $r_1 = 100$, $r_2 = 40$, $r_3 = 30$, $b = 10$, $c = 200$, $T = 20$, and $h(y) = b(y-c)^+$. Figure 1 illustrates the opportunity cost of accepting a class-3 request, $\Delta_3 v_t(x_1, x_2, x_3)$, for $t = 16$, over the state variable x_2 (left) when $x_1 = 4$, $x_3 = 1$ and over x_3 (right) when $x_1 = 4$, $x_2 = 6$, respectively.

Figure 1 shows that the opportunity cost $\Delta_3 v(x_1, x_2, x_3)$ is not monotone over the state variables x_2 and x_3. This implies that booking limit policy is not optimal for this

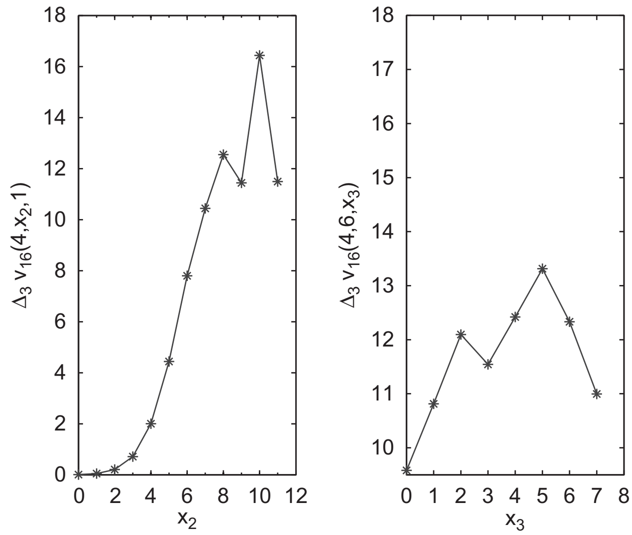

Figure 1 Nonmonotonicity for the 3-class problem.

problem. In the next two subsections, we restrict the general model along two different dimensions and explore the existence for booking limit policies. More specifically, in Section 4.2, we consider a two-class problem with general resource consumptions and in Section 4.3, we consider general *n*-class model but assume that resource consumptions are identically distributed.

4.2. Special case 1: two-class model with class-dependent resource consumptions

In the previous section, we show that opportunity cost is not necessarily monotone when $n \geqslant 3$. In the following lemma, we are able to show that it is indeed monotone when $n = 2$:

Lemma 2 *Assume that $n = 2$ and $\mathbf{x} = (x_1, x_2)$, then $\Delta_1 v_t(\mathbf{x}) \leqslant \Delta_1 v_t(\mathbf{x}+\mathbf{e}_2)$, and $\Delta_2 v_t(\mathbf{x}) \leqslant \Delta_2 v_t(\mathbf{x}+\mathbf{e}_1)$.*

The above result enables us to characterize the structure of the optimal control policy:

Theorem 2 *The optimal control policy of the two-class model is a state-dependent booking limit policy: it is optimal to accept a class-i request at time t if $x_j < S^*_{it}(x_i)$, and reject it otherwise, where $S^*_{1t}(x_1) = \min\{x_2 | \Delta_1 v_{t+1}(x_1, x_2) > r_1\}$ and $S^*_{2t}(x_2) = \min\{x_1 | \Delta_2 v_{t+1}(x_1, x_2) > r_2\}$.*

Now, we explore whether the state-dependent booking limits, that is, $S^*_{it}(x_i)$ are monotone over x_i. In order to prove it, we need additional property. Namely, opportunity cost should satisfy the following subconcavity condition: $\Delta_i v(\mathbf{x}+\mathbf{e}_i) \geqslant \Delta_i v(\mathbf{x}+\mathbf{e}_j)$ for $\forall i, j = 1, 2$, $i \neq j$. Unfortunately, we can construct a simple example to illustrate that the value function does not necessarily satisfy the above condition even for the deterministic resource consumptions.

Example 2 Consider the last period of a problem with two-class deterministic resource consumptions. Suppose that class-1 and class-2 consume $\mu_1=20$ and $\mu_2=10$ units of resources, and generate revenue of $r_1=200$ and $r_2=40$, respectively. Per-unit penalty cost is $b=20$, and the available capacity is $c=30$. The arrival probabilities of class-1 and class-2 are $\lambda_1=0.3$ and $\lambda_2=0.5$, respectively. Then, we can evaluate the value function at state $(x_1, x_2)=(0,0)$ in period T as follows: $v_T(0,0)=\lambda_1 r_1+\lambda_2 r_2=80$. Similarly, we can evaluate the value function at the following states: $v_T(0,1)=80$, $v_T(0,2)=20$ and $v_T(0,3)=0$. Since $v_T(0,0)-v_T(0,1)\leqslant v_T(0,1)-v_T(0,2)$ while $v_T(0,1)-v_T(0,2)\geqslant v_T(0,2)-v_T(0,3)$, the value function is neither componentwise concave nor convex on x_2. Note that $v_T(0,0)-v_T(0,1)=0\leqslant r_2$, $v_T(0,1)-v_T(0,2)=60\geqslant r_2$, $v_T(0,2)-v_T(0,3)=20\leqslant r_2$, it implies that in period T, it is optimal to accept class-2 arrival at state $(0,0)$, reject it at state $(0,1)$ and accept it again at state $(0,2)$.

The above example implies that the state-dependent booking limits $S^*_{it}(x_i)$ of the two-class random resource consumption model may not be monotone over x_i. Note that this is contrary to the unit demand case, in which we can easily show that whenever it is optimal to reject certain class at a lower state, it is also optimal to reject that class at a higher state. In the above example, the resource consumptions are assumed to be deterministic. We can also construct a 2-class example with random consumptions, in which booking limits are not monotone in state.

Example 3 Consider a 2-class problem with random resource consumption requirements A_1 and A_2, where A_1 and A_2 are normally distributed with $\mu_1=20$, $\sigma_1=6$ and $\mu_2=10$, $\sigma_2=2$, respectively. Assume that $\lambda_1=0.3$, $\lambda_2=0.5$, $r_1=100$, $r_2=40$, $b=10$, and $c=200$, $T=20$. The terminal value function $v_{T+1}(\mathbf{x})$ can be evaluated as follows:

$$v_{T+1}(\mathbf{x}) = -b\mathbf{E}\left[\sum_{i=1}^{n}\sum_{k=1}^{x_i} A_{ki} - c\right]^+ = -b\left[\sigma\phi\left(\frac{c-\mu}{\sigma}\right) - (c-\mu)\left(1-\Phi\left(\frac{c-\mu}{\sigma}\right)\right)\right],$$

where $\mu=x_1\mu_1+x_2\mu_2$, $\sigma=\sqrt{x_1\sigma_1^2+x_2\sigma_2^2}$, $\phi(\cdot)$, $\Phi(\cdot)$ are pdf and cdf of standard normal distribution respectively.

The optimal control policies are illustrated in Figure 2. From Figures 2(a) and 2(b), we can see that the optimal control policies for class-1 and class-2 are of booking limit type. That is, for i, $j=1,2$ and $i\neq j$, there exists a threshold $S^*_{it}(x_i)$ for any given x_i, such that it is optimal to accept class-i request if $x_j<S^*_{it}(x_i)$, and reject otherwise. However, as shown in Figure 2(b), for instance, $S^*_{2t}(6)=5$, $S^*_{2t}(7)=7$, $S^*_{2t}(8)=4$, from which we can see that the booking limit for class-2, $S^*_{2t}(x_2)$, is not monotone in x_2. Non-monotonicity of $S^*_{2t}(x_2)$ means that it can be optimal to accept class 2 at a higher state while reject it at a lower state.

In Figure 3, we also explore the mean and variance effects of the random resource consumptions on the total expected profit. Everything else being equal, the optimal total expected profit is decreasing in mean and variance, which supports the results in Theorem 1.

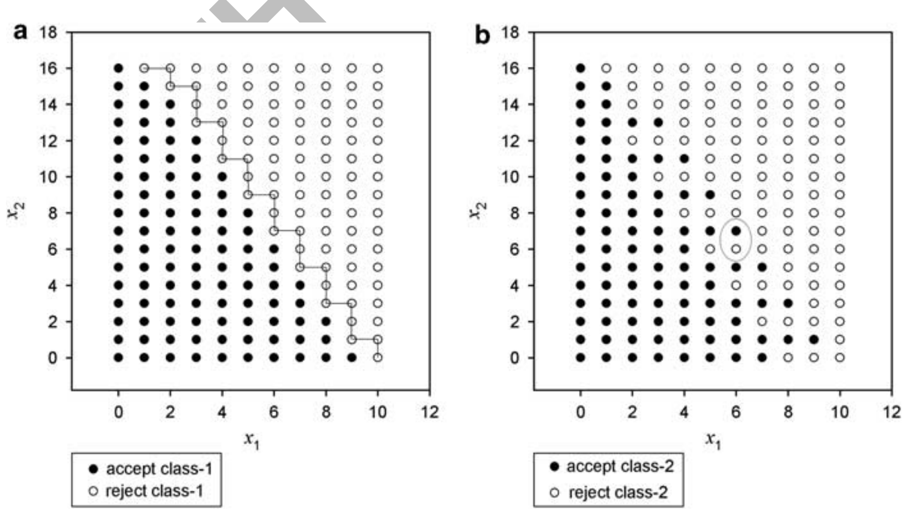

Figure 2 Optimal control policy for the two-class demand model with random resource consumptions. (a) Optimal control policy for class-1 at $t=16$. (b) Optimal control policy for class-2 at $t=16$.

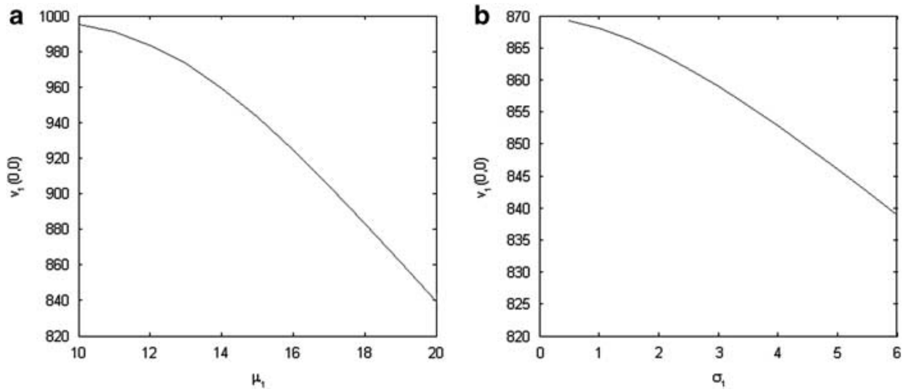

Figure 3 The effect of random resource consumptions on the optimal expected total profit. (a) Optimal expected total profit *versus* μ_1. (b) Optimal expected total profit *versus* σ_1. In Figure 3(a), we assume that $\sigma_1 = 6$, $(\mu_2, \sigma_2) = (10, 2)$; and in Figure 3(b), we assume that $\mu_1 = 20$, $(\mu_2, \sigma_2) = (10, 2)$.

4.3. Special case 2: n-class model with class-independent resource consumptions

In this section, we consider general n-class model, but assume that resource consumption distributions are identical. The analysis of this special case enables us to develop an efficient solution approach in the next section.

Note that when the resource consumption distribution is the same for all classes, that is, $F_i(\cdot) = F(\cdot)$ for all i, it suffices to record only the total number of reservations $x = \sum_{i=1}^{n} x_i$ as the state variable. Let $\tilde{v}_t(x)$ be the maximum total expected profit given state x at time t. Given t, x, the dynamic programming optimality equation can be written as

$$\begin{aligned}\tilde{v}_t(x) &= \sum_{i=1}^{n} \lambda_i \max\{\tilde{v}_{t+1}(x+1) + r_i,\ \tilde{v}_{t+1}(x)\} + \lambda_0 \tilde{v}_{t+1}(x) \\ &= \max_{\mathbf{u} \in \{0,1\}^n} \sum_{i=1}^{n} \lambda_i u_i [r_i - \Delta\tilde{v}_{t+1}(x)] + \tilde{v}_{t+1}(x), \qquad (6)\end{aligned}$$

where $\Delta\tilde{v}_{t+1}(x) = \tilde{v}_{t+1}(x) - \tilde{v}_{t+1}(x+1)$. The boundary conditions are

$$\tilde{v}_{t+1}(x) = -\mathbf{E}[h(Y_x)] \qquad \forall x, \qquad (7)$$

where Y_x is the sum of x independent random variables each with distribution $F(\cdot)$.

We show that having class-independent resource consumption preserves monotonicity of the value function over state x and time t. We characterize those properties in the following lemma.

Lemma 3 *We have the following results regarding the value function* $\tilde{v}_t(x)$:

(a) $\Delta\tilde{v}_t(x) \leqslant \Delta\tilde{v}_t(x+1)$,
(b) $\Delta\tilde{v}_t(x) \geqslant \Delta\tilde{v}_{t+1}(x)$.

By Lemma 3, we can define $S^*_{it} = \min\{x | \Delta\tilde{v}_{t+1}(x) > r_i\}$ and characterize the optimal admission policy π^*which is specified as follows:

$$u_i^{\pi^*}(x, t) = \begin{cases} 1 & \text{if } x < S^*_{it}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3 *The optimal control policy of the class-independent resource consumption distribution is a booking limit policy: it is optimal to accept a class-i request at time t if* $x < S^*_{it}$, *and reject it otherwise. Furthermore, the threshold* S^*_{it} *has the following properties:*

(a) *If* $r_i \geqslant \cdots \geqslant r_n$, *then* S^*_{it} *is non-decreasing in i, that is,* $S^*_{1t} \geqslant \cdots \geqslant S^*_{nt}$,
(b) S^*_{it} *is non-decreasing in t, that is,* $S^*_{it} \geqslant S^*_{i,t-1}$.

Theorem 3 shows that the optimal booking limits are time-dependent and nested in classes. Therefore, the traditional deterministic unit demand reduces to a special case of our class-independent resource consumption, where A_i is equal to 1 with probability 1 for all $i = 1, \ldots, n$. Making use of these structures, in the next section, we will develop a computationally efficient heuristic to solve the general n-class model with random resource consumptions.

5. Heuristics

In order to both evaluate the impact of considering random resource consumptions on the profits and develop computationally efficient solution approaches, we introduce six heuristics. The first two specifically use the information regarding the uncertainty in resource consumptions, whereas the other four ignore it. Later, in Section 6, we will test their performances under various scenarios.

5.1. Class-independent heuristic

The main idea of class-independent (CI) heuristic is to scale problem parameters by transforming an n-class model into a class-independent resource consumption model, thus reducing the n-dimensional MDP formulation to a one-dimensional MDP.

An important consideration in CI heuristic is parameter scaling. We introduce a parameter scaling that captures the random resource consumption by the first and second moments of the distribution. We scale the parameters $(\lambda_i, \mu_i, \sigma_i, r_i)$ of the general n-class model (2)–(3) with class-dependent random resource consumptions to $(\lambda_i^s, \mu^s, \sigma^s, r_i^s)$ with a common distribution captured by (μ^s, σ^s) as follows:

$$\mu^s = \frac{\sum_{i=1}^{n} \lambda_i \mu_i}{\sum_{i=1}^{n} \lambda_i}, \quad \lambda_i^s = \lambda_i \frac{\mu_i}{\mu^s}, \quad \lambda_0^s = \lambda_0,$$

$$\sigma^s = \sqrt{\frac{\sum_{i=1}^{n} \lambda_i^2 \sigma_i^2}{\sum_{i=1}^{n} (\lambda_i^s)^2}}, \quad r_i^s = r_i \frac{\mu_i}{\mu^s}. \tag{8}$$

Note that after scaling, the total arrival probability per period, mean and variance of the total resource consumptions per period, and unit revenue contribution for each i remain the same, that is, $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \lambda_i^s$, $\sum_{i=1}^{n} \lambda_i \mu_i = \sum_{i=1}^{n} \lambda_i^s \mu^s$, $\sum_{i=1}^{n} (\lambda_i \sigma_i)^2 = \sum_{i=1}^{n} (\lambda_i^s \sigma^s)^2$, and $(r_i/\mu_i) = (r_i^s/\mu^s)$ for all i. Note that in this scaled problem, the total resource consumption can be found by taking the convolution of all accepted random resource consumptions where each one is iid with mean and standard deviation given by (μ^s, σ^s).

The CI heuristic is to control the n-dimensional MDP (3) by using the optimal policy of the following one-dimensional MDP, with value function $\hat{v}_t^{ci}(x)$:

$$\hat{v}_t^{ci}(x) = \sum_{i=1}^{n} \lambda_i^s \max\{\hat{v}_{t+1}^{ci}(x+1) + r_i^s,\ \hat{v}_{t+1}^{ci}(x)\} + \lambda_i^s \hat{v}_{t+1}^{ci}(x), \qquad t = 1, \ldots, T, \tag{9}$$

with the boundary conditions $\hat{v}_{T+1}^{ci}(x) = -\mathbf{E}[h(Y_x^s)]$ for all x. By Lemma 3, we can define $S_{it}^{ci} = \min\{x | \Delta \hat{v}_{t+1}^{ci}(x) = \hat{v}_{t+1}^{ci}(x) - \hat{v}_{t+1}^{ci}(x+1) > r_i\}$ and characterize the control policy π^{ci} by

$$u_i^{ci}(x, t) = \begin{cases} 1 & \text{if } x < S_{it}^{ci}, \\ 0 & \text{otherwise.} \end{cases}$$

The control policy for the CI heuristic satisfies the properties of Theorem 3 with the nested structure and time monotonicity, and therefore, it is easy to compute and store. Let $v_t^{ci}(x)$ be the expected total profit from CI heuristic.

5.2. Decomposition heuristic

The decomposition (DC) heuristic decomposes the n-dimensional MDP into a collection of one-dimensional MDPs. We begin with a deterministic mathematical programming formulation, which treats both demands and resource consumptions as deterministic quantities by replacing random variables with their means. Let D_i be the total (random) number of class-i requests for the entire booking horizon. Clearly, $\mathbf{E}[D_i] = \lambda_i T$. Suppose the penalty function is given by $h(y) = -b(y-c)^+$. The deterministic mathematical programming formulation is given by

$$(\mathbf{NLP}) \quad \max \sum_{i=1}^{n} r_i x_i - b\left(\sum_{i=1}^{n} \mu_i x_i - c\right)^+$$
$$\text{s.t.} \quad 0 \leqslant x_i \leqslant E[D_i] \qquad \forall i.$$

Note that the above formulation (**NLP**) is a non-linear program; but it can be transformed into a linear program given as follows:

$$\begin{aligned} (\mathbf{DLP}) \quad & \max \sum_{i=1}^{n} r_i x_i - by \\ & \text{s.t.} \quad x_i \leqslant \mathbf{E}[D_i] \qquad \forall i, \\ & \qquad \sum_{i=1}^{n} \mu_i x_i - y \leqslant c, \\ & \qquad x_i \geqslant 0 \qquad \forall i, \\ & \qquad y \geqslant 0. \end{aligned} \tag{10}$$

Let $\gamma \geqslant 0$ be the shadow price of the capacity constraint (10), which captures the marginal value of capacity, and $(x_1^*, \ldots, x_n^*)$ be the optimal solutions of deterministic linear programming (DLP).

We now decompose the n-class MDP into n single-class programs, one for each class. When calculating the control policy for class i, we account for the 'average' value from all other classes obtained from the DLP shadow prices. Let $v_{t,i}^{dc}(x_i)$ be the value function that will be used to compute the control policy for class-i. Thus, for each i, we approximate the value function of (3) by

$$v_t(\mathbf{x}) \approx v_{t,i}^{dc}(x_i) + \sum_{j \neq i} \gamma_j x_j, \tag{11}$$

where $\gamma_j = \gamma \mathbf{E}[A_j]$. Since γ is the shadow price of capacity constraint, γ_j represents the approximate value of a class-j booking. Therefore, $\gamma_j x_j$ represents the value of all class-j bookings. The equation above effectively decomposes the value $v_t(\mathbf{x})$ into two components, the approximate value of class-i bookings $v_{t,i}^{dc}(x_i)$ and the approximate value of all other bookings. It should be noted that $v_{t,i}^{dc}(x_i)$ is an arbitrary function, while the value of all other bookings is approximated by a linear function. We would like to point out that a similar approximation architecture was used previously in network revenue management context; see, for example, Liu and van Ryzin (2008). Plugging (11) into (3), we have

$$\begin{aligned} v^{dc}_{t,i}(x_i) + \sum_{j\neq i}\gamma_j x_j = \lambda_i \max\Bigg\{ & v^{dc}_{t+1,i}(x_i+1) + \sum_{j\neq i}\gamma_j x_j \\ & + r_i,\ v^{dc}_{t+1,i}(x_i) + \sum_{j\neq i}\gamma_j x_j \Bigg\} \\ & \times \sum_{j\neq i}\lambda_j \max\Bigg\{ v^{dc}_{t+1,i}(x_i) + \sum_{k\neq i,j}\gamma_k x_k \\ & + \gamma_j(x_j+1),\ v^{dc}_{t+1,i}(x_i) + \sum_{j\neq i}\gamma_j x_j \Bigg\} \\ & + \lambda_0\left(v^{dc}_{t+1,i}(x_i) + \sum_{j\neq i}\gamma_j x_j \right). \end{aligned}$$

After some manipulations, the dynamic programming equation above can be reduced to

$$\begin{aligned} v^{dc}_{t,i}(x_i) = \lambda_i \max\left\{ v^{dc}_{t+1,i}(x_i+1) + r_i,\ v^{dc}_{t+1,i}(x_i)\right\} \\ + \sum_{j\neq i}\gamma_j\lambda_j + (1-\lambda_i)v^{dc}_{t+1,i}(x_i) \qquad \forall t,\ i. \end{aligned} \tag{12}$$

with the boundary condition

$$v^{dc}_{T+1,i}(x_i) = -\mathbf{E}\left[h\left(\sum_{k=1}^{x_i} A_{ki} + \sum_{j\neq i}\sum_{k=1}^{x_j^*} A_{ki} \right)\right]. \tag{13}$$

In particular, if $h(\cdot)$ is given by (1), then

$$v^{dc}_{T+1,i}(x_i) = -b\mathbf{E}\left[\sum_{k=1}^{x_i} A_{ki} + \sum_{j\neq i}\sum_{k=1}^{x_j^*} A_{ki} - c \right]^+. \tag{14}$$

It can be shown that the decomposed MDP (12)–(13) has the monotonicity properties that are characterized below.

Lemma 4 *We have the following results regarding the value function* $v^{dc}_{t,i}(x_i)$:

(a) $\Delta v^{dc}_{t,i}(x_i) \leqslant \Delta v^{dc}_{t,i}(x_i+1)$,

(b) $\Delta v^{dc}_{t,i}(x_i) \geqslant \Delta v^{dc}_{t+1,i}(x_i)$.

By Lemma 4, we obtain a state-dependent booking limit policy for each class i via the dynamic program (12)–(13). Let $S^{dc}_{it}(x_i) = \min\{x|\Delta v^{dc}_{t,i}(x_i) > r_i\}$. The control policy π^{dc} is given by

$$u^{dc}_i(\mathbf{x},\ t) = \begin{cases} 1 & \text{if } x < S^{dc}_{it}(x_i), \\ 0 & \text{otherwise.} \end{cases}$$

Finally, let $v^{dc}_t(x)$ be the expected total profit from DC heuristic.

5.3. Deterministic consumption heuristic

Another way to reduce the complexity of the problem is to consider an approximation based on deterministic consumption. In particular, we assume that, upon arrival, each class-i demand consumes exactly $\mu_i = \mathbf{E}[A_i]$ units of the resources. Let $\bar{v}_t(\mathbf{x})$ be the maximum expected profit of deterministic resource consumption model from period t onwards given state $\mathbf{x}$. The optimality equation is given by

$$\begin{aligned} \bar{v}_t(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \max\{\bar{v}_{t+1}(\mathbf{x}+\mathbf{e}_i) + r_i,\ \bar{v}_{t+1}(\mathbf{x})\} \\ + \lambda_0 \bar{v}_{t+1}(\mathbf{x}) \end{aligned} \tag{15}$$

with boundary conditions $\bar{v}_{T+1}(\mathbf{x}) = -h(\sum_{i=1}^{n} x_i\mu_i)$. Note that $\mathbf{E}[h(Y_\mathbf{x})] \geqslant h(\mathbf{E}[Y_\mathbf{x}]) = h(\sum_{i=1}^{n} x_i\mu_i)$ for convex function $h(\cdot)$. Following the same argument as Theorem 1(a), it follows that the stochastic resource consumption case achieves no higher optimal expected profit than that of the deterministic case, which is summarized in the following corollary.

Corollary 1 $v_t(\mathbf{x}) \leqslant \bar{v}_t(\mathbf{x})$, $\forall t$, $\mathbf{x}$.

Corollary 1 says that problems with uncertain resource consumptions achieve lower expected profits than the case with deterministic resource consumption. Assuming deterministic resource consumption, we have the following result which can be used to reduce the state space from n-dimensional to one-dimensional. The proof is straightforward and is omitted.

Lemma 5 $\bar{v}_t(\mathbf{x}) = \bar{v}_t(\tilde{\mathbf{x}})$ *for all* $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *that satisfy* $\sum_{i=1}^{n} x_i\mu_i = \sum_{i=1}^{n} \tilde{x}_i\mu_i$.

We construct a heuristic, which is based on the deterministic consumption, and we call it as DET heuristic. By Lemma 5, it suffices to record the total resource requirement $y = \sum_{i=1}^{n} x_i\mu_i$ as the state of DET heuristic. Let $\hat{v}^{det}_t(y)$ be the value function when the total resources required for all reservations at time t is y. The DET heuristic is to control the n-dimensional MDP (3) by using the optimal policy of the following one-dimensional MDP:

$$\bar{v}^{det}_t(y) = \sum_{i=1}^{n} \lambda_i \max\{\bar{v}^{det}_{t+1}(y+\mu_i) + r_i,\ \bar{v}^{det}_{t+1} v(y)\}, \tag{16}$$

with the boundary conditions $\hat{v}^{det}_{T+1}(y) = -h(y)$ for all y.

Note that $v_1(0,\ldots,0) \leqslant \bar{v}_1(0,\ldots,0) = \hat{v}^{det}_1(0)$. Thus, DET heuristic provides an upper bound for the n-class MDP formulation with random resource consumptions. Under the DET heuristic, the control policy π^{det} is given by

$$u^{det}_i(y,\ t) = \begin{cases} 1 & \text{if } r_i \geqslant v^{det}_{t+1}(y) - v^{det}_{t+1}(y+\mu_i) \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, let $v^{det}_t(x)$ be the expected total profit from DET heuristic.

5.4. Other heuristics

Suppose that the optimal solution of (**DLP**) is given by $\mathbf{x}^*$ and the shadow price of the capacity constraint is γ. Then, control policy of DLP heuristic π^{dlp} is given by

$$u_i^{dlp}(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{k=1}^{n} x_k\mu_k + \mu_i \leqslant c \text{ and } x_i < x_i^*, \\ 0 & \text{otherwise.} \end{cases}$$

The control policy of bid-price (BP) heuristic π^{bp} is given by

$$u_i^{bp}(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{k=1}^{n} x_k\mu_k + \mu_i \leqslant c \text{ and } r_i \geqslant \gamma\mu_i, \\ 0 & \text{otherwise.} \end{cases}$$

We also consider the first-come first-serve (FCFS) policy, for which the control policy π^{fcfs} is given by

$$u_i^{fcfs}(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{k=1}^{n} x_k\mu_k + \mu_i \leqslant c \\ 0, & \text{otherwise.} \end{cases}$$

Denote the expected profit of DLP, BP and FCFS heuristics at period t given state x as $v_t^{dlp}(x)$, $v_t^{bp}(x)$, and $v_t^{fcfs}(x)$, respectively.

6. Numerical results and managerial insights

In this section, we conduct numerical experiments to study the relative performance of the different heuristics. We design three sets of examples. The first two sets have two customer classes, and the third set has eight customer classes. Since it is feasible to solve for the optimal strategy for two-class problems, considering two class cases allows us to compare the performance of different heuristics to the optimal policy. When there are eight classes, the corresponding dynamic programming formulation has eight-dimensional state space, and therefore is difficult to solve exactly. Therefore, we compare the performance of the different heuristic policies to an upper bound. Define the relative percentage differences of the expected revenue for a heuristic policy from the optimal expected value and upper bound of the optimal expected value, respectively, as follows:

$$\beta_h = \frac{v_1(0,\ldots,0) - v_1^h(0,\ldots,0)}{v_1(0,\ldots,0)} \times 100\%,$$

$$\bar{\beta}_h = \frac{\bar{v}_1(0,\ldots,0) - v_1^h(0,\ldots,0)}{\bar{v}_1(0,\ldots,0)} \times 100\%,$$

where $h \in \{ci, dc, det, dip, bp, fcfs\}$.

Within each numerical set, we vary three factors: the capacity/demand ratio, the penalty cost/average unit revenue ratio, and the coefficient of variance for the consumption of each class. The average per unit revenue and the capacity/demand ratio can be defined as $\bar{r} = \sum_{i=1}^{n} \lambda_i r_i T/d$ and c/d, respectively, where $d = T\sum_{i=1}^{n}(\lambda_i\mu_i)$ stands for total potential expected resource consumption. We consider three different values for c/d: 0.5, 0.75, and 1. Note that we focus on cases where the capacity is relatively tight because the value of optimal control is particularly high in such cases. The ratio of penalty cost-to-average unit revenue is given by $b/\bar{r}$, for which we also consider three levels with values 0.9, 1.0, and 1.1, respectively.

We use Lognormal distribution to model the random resource consumptions, ie, A_i are independent and lognormally distributed with mean μ_i and standard deviation σ_i, $i = 1, \ldots, n$. In each instance, we set μ_i and σ_i such that coefficient of variation, $cv_i = \sigma_i/\mu_i$, is same for all classes, ie, $cv_i = cv$ for all $i = 1, \ldots, n$. Also, by varying (μ_i, σ_i), we consider four different coefficient of variations with values $cv \in \{0.5, 1, 2, 4\}$ to represent the different degrees of consumption uncertainty. Therefore, in each test set, there are $3 \times 3 \times 4 = 36$ scenarios.

6.1. Two class with class-independent random resource consumption

We first consider a set of two class problems with the same random resource consumption for each class, ie, $n = 2$, $\mu_1 = \mu_2 = \mu = 5$ and $\sigma_1 = \sigma_2 = \sigma$. We vary σ so that $cv = \sigma/\mu$ takes 0.5, 1, 2 and 4. The other problem parameters are as follows: $T = 50$, $\lambda_1 = 0.2$, $\lambda_2 = 0.3$, $r_1 = 50$, $r_2 = 40$. Then, $d = T\sum_{i=1}^{2}(\lambda_i\mu_i) = 125$ and the average unit revenue is $\bar{r} = 8.80$. The results are reported in Table 1.

Note that since the resource consumption is the same for the two classes, CI heuristic is optimal and, therefore, its performance gap is not reported in the table. The DC heuristic performs the best on average among all five heuristics with an average performance gap of 0.82% and a maximum percentage gap of 5.74%. The performance of the DET heuristic is ranked second with an average percentage gap of 3.61% and a maximum percentage gap of 13.35%. The DET heuristic works the best when the demand variation is small, ie, when the coefficient of variation is no greater than 1. The performance of DET degrades as the demand variance and/or the penalty cost increase. The profit loss from the deterministic heuristic control can reach as high as 13.35% when $cv = 4$ and $b/\bar{r} = 1$. On the other hand, the performance of DLP, BP and FCFS heuristics can be very poor with a percentage gap as high as 33% when coefficient of variation is high ($cv = 4$).

In general, the performance gap of all heuristics is increasing in the tightness of the capacity. This observation is intuitive as optimal control policy is most valuable when the capacity is insufficient to satisfy the total expected demand. We also observe very different performance across the different heuristics, with DC and DET heuristics performing the best among all five heuristics. The performance gap is increasing in the penalty cost for DC heuristic, whereas it is decreasing in the penalty cost for

Table 1 Percentage gap from the optimal expected revenue for two-class problems with class-independent consumption distribution

cv	*b/r*	*c/d*=0.5					*c/d*=0.75					*c/d*=1				
		β_{dc}	β_{det}	β_{dlp}	β_{bidp}	β_{fcfs}	β_{dc}	β_{det}	β_{dlp}	β_{bidp}	β_{fcfs}	β_{dc}	β_{det}	β_{dlp}	β_{bidp}	β_{fcfs}
	0.9	0.04	0.00	16.18	16.18	16.18	0.02	0.00	9.65	9.33	9.33	0.00	0.00	9.24	1.88	1.88
0.5	1	3.95	0.95	16.78	13.74	13.74	1.82	1.03	12.58	7.70	7.70	0.51	0.53	8.70	1.50	1.50
	1.1	5.74	0.60	15.35	12.49	12.49	2.73	0.72	11.45	6.69	6.69	0.91	0.38	8.23	1.20	1.20
	0.9	0.17	0.00	20.32	20.32	20.32	0.02	0.00	12.37	12.07	12.07	0.00	0.00	9.24	2.50	2.50
1	1	2.17	3.20	18.66	17.01	17.01	0.98	2.89	13.96	10.06	10.06	0.08	1.00	8.62	2.11	2.11
	1.1	2.78	2.16	16.45	15.14	15.14	1.30	2.12	12.26	8.55	8.55	0.13	0.82	8.02	1.76	1.76
	0.9	0.25	0.01	26.61	26.61	26.61	0.02	0.00	15.72	15.43	15.43	0.00	0.00	9.68	3.06	3.06
2	1	0.82	7.71	23.15	22.48	22.48	0.24	5.75	16.83	13.48	13.48	0.00	1.49	9.07	2.73	2.73
	1.1	2.04	5.42	19.80	19.61	19.61	0.38	4.60	14.70	11.60	11.60	0.02	1.33	8.45	2.38	2.38
	0.9	0.26	0.00	32.97	32.96	32.96	0.00	0.01	18.65	18.33	18.33	0.00	0.01	10.36	3.50	3.50
4	1	0.87	13.35	28.83	28.92	28.92	0.04	8.29	19.57	16.67	16.67	0.00	1.85	9.79	3.20	3.20
	1.1	1.07	9.89	24.55	25.30	25.30	0.21	7.17	17.44	14.88	14.88	0.00	1.74	9.18	2.87	2.87
Minimum		0.04	0.00	15.35	12.49	12.49	0.00	0.00	9.65	6.69	6.69	0.00	0.00	8.02	1.20	1.20
Average		1.68	3.61	21.64	20.90	20.90	0.65	2.72	14.60	12.06	12.06	0.14	0.76	9.05	2.39	2.39
Maximum		5.74	13.35	32.97	32.96	32.96	2.73	8.29	19.57	18.33	18.33	0.91	1.85	10.36	3.50	3.50

Note: $T=50$, $\lambda_1=0.2$, $\lambda_2=0.3$, $n=50$, $r_2=40$, $\mu_1=\mu_2=\mu=5$.

Table 2 Parameters for eight-class problems

	Class-1	*Class*-2	*Class*-3	*Class*-4	*Class*-5	*Class*-6	*Class*-7	*Class*-8
λ_i	0.1	0.1	0.1	0.1	0.05	0.05	0.05	0.05
μ_i	16	14	12	10	8	6	4	2
r_i	115.2	106.4	96	84	70.4	55.2	38.4	20

DLP, BP and FCFS heuristics. The impact of penalty cost for DET heuristic is, in general, non-monotonic. The performance of DC heuristic is decreasing in the coefficient of variation, while the performance of the other four heuristics increases in the coefficient of variation (Table 1).

Since DET, DLP, BP and FCFS heuristics do not capture the randomness in resource consumption and/or demand, this observation coincides with our expectation.

Overall, our results show that DC heuristic is a viable control strategy with relatively small performance gap, while the other heuristics can bring significant profit loss. This conclusion also holds for the other two sets of problems that we introduce next.

6.2. Two-class with class-dependent random resource consumption

We now turn our attention to the second set of two-class problems with different random resource consumptions, where $n=2$, $\mu_1=10$ and $\mu_2=5$. We take $T=50$, $\lambda_1=0.3$, $\lambda_2=0.2$, $r_1=80$, and $r_2=50$. Then, $d=200$, and the average per unit revenue is $\bar{r}=8.5$. The results are reported in Table 3.

First, we note that since the resource consumptions differ across the two classes, CI heuristic is no longer optimal. Therefore, we report the results of CI heuristic. We again compare all heuristics against the optimal policy. Overall, CI heuristic and DC heuristic work better than all the other heuristics, with CI heuristic slightly better than DC heuristic. The average and maximum profit gap of the CI heuristic are 0.41 and 3.99%, respectively, while those of the DC heuristic are 0.50 and 4.82%, respectively. When the demand variations are low, DET heuristic works well regardless of the penalty cost. However, if the demand is highly variable, a maximum loss of 16.32% incurs when $cv=4$ and $b/\bar{r}=1$. We also observe a high performance gap when the capacity is tighter. As $b/\bar{r}$ increases, the performance gaps of DLP, BP, and FCFS heuristics decrease. Our overall conclusions that we draw from this problem set are similar to those in the previous subsection.

6.3. An eight-class example with class-dependent random resource consumption

Now, we consider a set of relatively large problems with eight classes of demand, where $n=8$, and λ_i, μ_i and r_i for all $i=1,\ldots,8$, are reported in Table 2. Also, in each problem instance, we change σ_i so that coefficient of variation for each class i is kept same, ie, $cv_i=cv$ for all $i=1,\ldots,8$. The total number of periods is $T=50$. The average per unit revenue is $\bar{r}=7.96$ and $d=310$.

Since it is computationally very costly to evaluate different policies due to the 'curse of dimensionality', we use simulation to estimate the expected revenue for each heuristic. We randomly generate arrivals of the eight classes according to the arrival probabilities in Table 2. We run 2000 simulations for each problem instance. In

Table 3 Percentage gap from the optimal expected revenue for two-class problems with class-dependent consumption distributions

cv	b/r	$c/d=0.5$						$c/d=0.75$						$c/d=1$					
		β_{ci}	β_{dc}	β_{det}	β_{dmp}	β_{bidp}	β_{fcfs}	β_{ci}	β_{dc}	β_{det}	β_{dmp}	β_{bidp}	β_{fcfs}	β_{ci}	β_{dc}	β_{det}	β_{dmp}	β_{bidp}	β_{fcfs}
	0.9	0.00	0.00	0.00	13.58	13.35	13.35	0.00	0.00	0.00	7.98	6.80	6.80	0.00	0.00	0.00	8.22	2.07	2.07
0.5	1	3.99	2.62	1.78	10.37	9.05	9.05	0.80	0.78	1.68	7.61	4.82	4.82	0.04	0.29	0.69	7.61	1.61	1.61
	1.1	2.13	4.82	1.19	8.89	7.85	7.85	0.45	1.47	1.18	6.49	3.86	3.86	0.11	0.58	0.51	5.62	1.26	1.26
	0.9	0.00	0.00	0.00	17.94	17.71	17.71	0.00	0.00	0.00	10.85	9.67	9.67	0.00	0.00	0.00	8.46	2.74	2.74
1	1	2.64	1.09	4.80	13.48	12.95	12.95	0.24	0.45	4.01	10.04	7.53	7.53	0.01	0.03	1.28	7.83	2.31	2.31
	1.1	0.51	1.48	3.34	11.10	10.96	10.96	0.43	1.25	2.95	8.33	5.99	5.99	0.03	0.15	1.05	5.85	1.89	1.89
	0.9	0.00	0.03	0.00	24.25	24.02	24.02	0.00	0.00	0.00	14.26	13.00	13.00	0.00	0.00	0.00	9.10	3.39	3.39
2	1	0.97	0.33	10.10	18.95	19.05	19.05	0.03	0.04	7.16	13.52	11.10	11.10	0.00	0.01	1.85	8.51	3.02	3.02
	1.1	0.52	0.92	7.15	15.23	15.82	15.82	0.23	0.41	5.74	11.48	9.23	9.23	0.00	0.02	1.65	6.52	2.63	2.63
	0.9	0.01	0.05	0.01	30.46	30.22	30.22	0.00	0.00	0.00	17.23	15.82	15.82	0.00	0.00	0.00	9.93	3.99	3.99
4	1	0.24	0.42	16.32	25.26	25.86	25.86	0.00	0.02	9.83	16.77	14.21	14.21	0.00	0.00	2.28	9.40	3.66	3.66
	1.1	1.38	0.57	11.90	20.30	21.53	21.53	0.05	0.14	8.49	14.86	12.47	12.47	0.00	0.00	2.12	7.31	3.31	3.31
Minimum		0.00	0.00	0.00	8.89	7.85	7.85	0.00	0.00	0.00	6.49	3.86	3.86	0.00	0.00	0.00	5.62	1.26	1.26
Average		1.03	1.03	4.72	17.49	17.36	17.36	0.18	0.38	3.42	11.62	9.54	9.54	0.02	0.09	0.95	7.86	2.66	2.66
Maximum		3.99	4.82	16.32	30.46	30.22	30.22	0.80	1.47	9.83	17.23	15.82	15.82	0.11	0.58	2.28	9.93	3.99	3.99

Note. $T=50$, $\lambda_1=0.3$, $\lambda_2=0.2$, $r_1=80$, $r_2=50$, $\mu_1=10$, $\mu_2=5$.

Table 4 Percentage gap from an upper bound for eight-class problems with class-dependent consumption distributions

cv	b/r	$c/d=0.5$						$c/d=0.75$						$c/d=1$					
		$\bar{\beta}_{ci}$	$\bar{\beta}_{dc}$	$\bar{\beta}_{det}$	$\bar{\beta}_{dlp}$	$\bar{\beta}_{bidp}$	$\bar{\beta}_{fcfs}$	$\bar{\beta}_{ci}$	$\bar{\beta}_{dc}$	$\bar{\beta}_{det}$	$\bar{\beta}_{dlp}$	$\bar{\beta}_{bidp}$	$\bar{\beta}_{fcfs}$	$\bar{\beta}_{ci}$	$\bar{\beta}_{dc}$	$\bar{\beta}_{det}$	$\bar{\beta}_{dlp}$	$\bar{\beta}_{bidp}$	$\bar{\beta}_{fcfs}$
	0.9	0.04	0.04	0.05	11.95	11.83	11.83	0.23	0.23	0.23	6.78	5.79	5.79	0.06	0.06	0.06	16.10	15.98	15.98
0.5	1	3.33	4.26	1.24	9.54	8.68	9.41	2.55	4.43	1.78	6.07	7.58	5.12	2.98	3.04	4.13	14.31	12.28	14.18
	1.1	12.33	8.84	3.25	9.62	32.97	9.49	5.42	5.59	3.24	6.24	7.54	5.34	8.77	6.40	7.80	14.91	33.78	14.77
	0.9	0.06	0.06	0.06	16.10	15.98	15.98	1.39	1.41	1.39	10.53	9.42	9.42	3.65	3.79	3.65	14.61	5.48	5.48
1	1	2.98	3.04	4.13	14.31	12.28	14.18	3.39	4.39	5.30	10.27	11.16	9.21	4.34	4.59	5.07	14.61	5.85	5.85
	1.1	8.77	6.40	7.80	14.91	33.78	14.77	6.16	7.16	7.61	10.84	11.46	9.84	5.26	5.80	6.19	14.79	6.41	6.41
	0.9	0.56	0.57	0.53	22.69	22.58	22.58	4.69	4.94	4.70	16.67	15.66	15.66	8.95	9.82	8.94	19.46	11.16	11.16
2	1	4.85	5.78	10.06	21.87	18.83	21.76	7.19	7.79	11.73	17.16	17.48	16.20	10.23	11.40	11.32	20.03	12.16	12.17
	1.1	10.48	12.41	15.58	23.29	38.19	23.17	10.42	12.38	15.01	18.45	18.42	17.56	11.71	13.27	13.13	20.77	13.36	13.36
	0.9	2.66	2.73	2.67	30.65	30.49	30.50	10.50	11.73	10.50	24.29	23.36	23.36	16.58	19.86	16.58	26.85	18.88	18.88
4	1	8.60	9.75	18.14	31.04	26.93	30.88	13.74	15.98	20.05	25.75	25.30	24.87	18.68	22.96	19.87	28.21	20.73	20.73
	1.1	15.77	20.80	25.03	33.44	44.12	33.29	7.90	21.95	24.37	27.95	27.11	27.14	20.98	26.54	22.52	29.73	22.76	22.77
Minimum		0.04	0.04	0.05	9.54	8.68	9.41	0.23	0.23	0.23	6.07	5.79	5.12	0.06	0.06	0.06	14.31	5.48	5.48
Average		5.87	6.22	7.38	19.95	24.72	19.82	6.96	8.16	8.83	15.08	15.02	14.13	9.35	10.63	9.94	19.53	14.90	13.48
Maximum		15.77	20.80	25.03	33.44	44.12	33.29	17.90	21.95	24.37	27.95	27.11	27.14	20.98	26.54	22.52	29.73	33.78	22.77

almost all cases, the simulation error is less than 0.5%, which is much smaller than the relative difference among different policies. The relative performance gap of each heuristic compared with the upper bound is given in Table 4.

The CI heuristic is again the best on average among all heuristics. The performance of DC heuristic ranks second when the tightness of the capacity is high or medium. The DET heuristic works well only when $cv=0.5$ and the tightness of the capacity is medium. As the coefficient of variation, the tightness of capacity, and the penalty cost increase, the CI heuristic improves over the DET heuristic. For example, when $cv=4$, $c/d=0.5$ and $b/\bar{r}\geqslant 1.1$, the CI heuristic generates 10% more revenue than the DET heuristic.

Our results in this section confirm once again that CI heuristic performs well under a variety of parameter settings. DC heuristic also performs well in general, but seems to be slightly worse than CI heuristic. DET heuristic is the best among all other heuristics, but is still dominated by both CI and DC heuristics. Therefore, our results point to the importance of considering random resource consumption in control policies. Indeed, ignoring random resource consumption can result in huge loss in profits, especially when the coefficient of variation is high.

7. Conclusions

In this paper, we build a dynamic programming model to analyze the impact of random resource consumptions on optimal revenue management decisions. We characterize the structure of optimal policies for the two-class case and show that the optimal booking decision is of threshold type (known as *booking limit* policy). However, we show via counterexamples that booking limit policy may not be generally optimal when the number of customer classes exceeds two. This result necessitates us to explore computationally efficient heuristic policies that specifically take into account the random resource consumptions.

Along these lines, we develop two heuristics that take quite different approaches to break the curse of dimensionality. The first heuristic (CI) pools random consumption distributions of all the classes into a common distribution by scaling their first and second moments, and then solves a revenue management problem where random resource consumptions follow this common distribution. The second heuristic (DC), instead, decomposes the multi-class problem into several independent single-class problems, all of which are linked only by a common terminal condition. In addition to the above two, we consider three other heuristics (DET, BP, DLP) commonly utilized in the literature that ignore uncertainties in demand arrivals and/or resource consumptions. We also consider FCFS as a benchmark. We conduct an extensive numerical study to compare the performance of CI and DC with that of DET, BP, DLP, and FCFS under various scenarios.

We show that CI performs the best among all the heuristics. Indeed, we can analytically prove that CI is optimal when the distribution function that governs uncertainty in resource consumption is the same for all the classes. The next best-performing heuristic is DC. We also identify three conditions under which CI and DC outperform the others. Specifically, they are coefficient of variation, demand-to-capacity ratio and penalty cost-to-unit revenue ratio, representing the degree of randomness in the resource consumption, the capacity utilization, and relative ratio of underage cost to overage cost, respectively. In general, the performance of all heuristics worsens as all three factors increase. In the mean time, as all these factors increase, the performance gap between CI and DC at one side and the four other heuristics on the other side widens.

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Appendix

Proof of Theorem 1 We only show the proof for convex order. The proof for usual stochastic order follows similarly. (a) We first show that $v_{T+1}(\tilde{\mathbf{x}}) \leqslant v_{T+1}(\mathbf{x})$ if $Y_{\tilde{\mathbf{x}}} \geqslant_{cx} Y_{\mathbf{x}}$. Note that

$$v_{T+1}(\tilde{\mathbf{x}}) = -\mathbf{E}[h(Y_{\tilde{\mathbf{x}}})] \leqslant -\mathbf{E}[h(Y_{\mathbf{x}})] = v_{T+1}(\mathbf{x}),$$

where the inequality follows from the facts that $Y_{\tilde{\mathbf{x}}} \geqslant_{cx} Y_{\mathbf{x}}$ and $h(\cdot)$ is convex.

Assume that $v_{t+1}(\tilde{\mathbf{x}}) \leqslant v_{t+1}(\mathbf{x})$ if $Y_{\tilde{\mathbf{x}}} \geqslant_{cx} Y_x$, and we need to show that $v_t(\tilde{\mathbf{x}}) \leqslant v_t(\mathbf{x})$. By (5), the key is to show that $T_k v_{t+1}(\tilde{\mathbf{x}}) \leqslant T_k v_{t+1}(\mathbf{x})$, ie,

$$\max\{v_{t+1}(\tilde{\mathbf{x}} + \mathbf{e}_k) + r_k,\ v_{t+1}(\tilde{\mathbf{x}})\} \leqslant \max\{v_{t+1}(\mathbf{x} + \mathbf{e}_k) + r_k,\ v_{t+1}(\mathbf{x})\} \qquad (A.1)$$

Note that $Y_{\tilde{\mathbf{x}}+\mathbf{e}_k} = Y_{\tilde{\mathbf{x}}} + A_k$ and $Y_{\mathbf{x}+\mathbf{e}_k} = Y_{\mathbf{x}} + A_k$, thus $Y_{\tilde{\mathbf{x}}+\mathbf{e}_k} \geqslant_{cx} Y_{\mathbf{x}+\mathbf{e}_k}$ by the closure property of convex order and $v_{t+1}(\tilde{\mathbf{x}} + \mathbf{e}_k) \leqslant v_{t+1}(\mathbf{x} + \mathbf{e}_k)$ by induction assumption. Therefore (A.1) holds true, ie, $T_k v_{t+1}(\tilde{\mathbf{x}}) \leqslant T_k v_{t+1}(\mathbf{x})$ given $Y_{\tilde{\mathbf{x}}} >_{cx} Y_{\mathbf{x}}$. It follows that

$$v_t(\tilde{\mathbf{x}}) - v_t(\mathbf{x}) = \sum_{k=1}^{n} \lambda_k [T_k v_{t+1}(\tilde{\mathbf{x}}) - T_k v_{t+1}(\mathbf{x})] + \lambda_0 [v_{t+1}(\tilde{\mathbf{x}}) - v_{t+1}(\mathbf{x})] \leqslant 0,$$

and we have shown that $v_t(\tilde{\mathbf{x}}) \leqslant v_t(\mathbf{x})$ if $Y_{\tilde{\mathbf{x}}} \geqslant_{cx} Y_{\mathbf{x}}$.
(b) The result follows from (a) immediately as $Y_{\mathbf{x}} + A_i \geqslant_{cx} Y_{\mathbf{x}} + A_j$ given $A_i \geqslant_{cx} A_j$. Therefore, $v_t(\mathbf{x} + \mathbf{e}_i) \leqslant v_t(\mathbf{x} + \mathbf{e}_j)$ and $\Delta_i v_t(\mathbf{x}) \geqslant \Delta_j v_t(\mathbf{x})$.
(c) Suppose $A_i \geqslant_{cx} A_j$ and $r_j \geqslant r_i$, that is, class-j is ranked higher than class-i. By (b), we have $\Delta_i v_t(\mathbf{x}) \geqslant \Delta_j v_t(\mathbf{x})$ if $A_i \geqslant_{cx} A_j$. Therefore the result follows directly from

$$r_j \geqslant r_i \geqslant \Delta_i v_t(\mathbf{x}) \geqslant \Delta_j v_t(\mathbf{x}),$$

that is, if it is optimal to accept a class-i request at state (x, t), then it is also optimal to accept a class-j request which is ranked higher than class-i. □

Proof of Lemma 2 First we use coupling arguments to prove that the terminal function $v_{T+1}(\mathbf{x}) = -\mathbf{E}[h(Y_{\mathbf{x}})]$ preserves submodularity for any nondecreasing and convex $h(\cdot)$.

Let $f(\mathbf{x}) = -v_{T+1}(\mathbf{x})$ represent the penalty cost given the state $\mathbf{x}$ at period $T+1$. Submodularity of $v_{T+1}(\mathbf{x})$ is equivalent to

$$f(\mathbf{x} + \mathbf{e}_1) - f(\mathbf{x}) \leqslant f(\mathbf{x} + \mathbf{e}_1 + \mathbf{e}_2) - f(\mathbf{x} + \mathbf{e}_2), \qquad (A.2)$$

which says that the marginal cost of accepting a class-1 request at state $\mathbf{x}$ is lower than that of at state $\mathbf{x} + \mathbf{e}_2$. Consider two systems with system 1 at state $\mathbf{x}$ and system 2 at state $\mathbf{x} + \mathbf{e}_2$; and the arrival processes and random resource requirements of the two systems are coupled. We compare the marginal cost of accepting a class-1 request for the two systems. There are three cases: (1) both the total resource requirement of accepting a class-1 demand at state $\mathbf{x}$ and that of accepting another class-1 at state $\mathbf{x} + \mathbf{e}_2$ do not exceed the capacity; (2) the total resource requirement of accepting a class-1 demand at state $\mathbf{x}$ is within the capacity, while that of accepting a class-1 at state $\mathbf{x} + \mathbf{e}_2$ exceeds the capacity; (3) both have exceeded the capacity. In the first case, the marginal costs of system 1 and system 2 are the same; while in the second and third case, the marginal cost of system 1 is less than that of system 2 for the linear cost function. Therefore (A.2) is true and submodularity holds for $v_{T+1}(\mathbf{x})$.

Now assume that $\Delta_1 v_{t+1}(x_1, x_2) \leqslant \Delta_1 v_{t+1}(x_1, x_2 + 1)$ and we need to show that submodularity holds for $v_t(x_1, x_2)$, which is essentially reduced to show that $\Delta_1 T_i v_{t+1}(x_1, x_2) \leqslant \Delta_1 T_i v_{t+1}(x_1, x_2 + 1)$, $i = 1, 2$. We will adopt an approach introduced by Zhuang and Li (Lemma 2, 2010) to prove the propagation of structural properties on the control operator T_i.

(a) $T_1 v_{t+1}(x_1, x_2) - T_1 v_{t+1}(x_1 + 1, x_2) \leqslant T_1 v_{t+1}(x_1, x_2 + 1) - T_1 v_{t+1}(x_1 + 1, x_2 + 1)$, ie,

$$\begin{aligned} &\max\{v_{t+1}(x_1 + 1,\ x_2) + r_1,\ v_{t+1}(x_1,\ x_2)\} \\ &\quad - \max\{v_{t+1}(x_1 + 2,\ x_2) + r_1,\ v_{t+1}(x_1 + 1,\ x_2)\} \\ &\quad \leqslant \max\{v_{t+1}(x_1 + 1,\ x_2 + 1) + r_1,\ v_{t+1}(x_1,\ x_2 + 1)\} \\ &\quad - \max\{v_{t+1}(x_1 + 2,\ x_2 + 1) + r_1,\ v_{t+1}(x_1 + 1,\ x_2 + 1)\}. \end{aligned} \qquad (A.3)$$

Since

$$\begin{aligned} &v_{t+1}(x_1 + 1,\ x_2) + r_1 - (v_{t+1}(x_1 + 2,\ x_2) + r_1) \\ &\quad \leqslant v_{t+1}(x_1 + 1,\ x_2 + 1) + r_1 - (v_{t+1}(x_1 + 2,\ x_2 + 1) + r_1), \\ &v_{t+1}(x_1 + 1,\ x_2) + r_1 - v_{t+1}(x_1 + 1,\ x_2) \\ &\quad \leqslant (=) v_{t+1}(x_1 + 1,\ x_2 + 1) + r_1 - v_{t+1}(x_1 + 1,\ x_2 + 1), \\ &v_{t+1}(x_1,\ x_2) - (v_{t+1}(x_1 + 2,\ x_2) + r_1\} \\ &\quad \leqslant v_{t+1}(x_1,\ x_2 + 1) - (v_{t+1}(x_1 + 2,\ x_2 + 1) + r_1), \\ &v_{t+1}(x_1,\ x_2) - v_{t+1}(x_1 + 1,\ x_2) \\ &\quad \leqslant v_{t+1}(x_1,\ x_2 + 1) - v_{t+1}(x_1 + 1,\ x_2 + 1), \end{aligned}$$

where the inequalities follow from submodularity of $v_{t+1}(x_1, x_2)$ that

$$\begin{aligned} &\Delta_1 v_{t+1}(x_1 + 1,\ x_2) \leqslant \Delta_1 v_{t+1}(x_1 + 1,\ x_2 + 1), \\ &\Delta_2 v_{t+1}(x_1,\ x_2) \leqslant \Delta_2 v_{t+1}(x_1 + 2,\ x_2), \\ &\Delta_1 v_{t+1}(x_1,\ x_2) \leqslant \Delta_1 v_{t+1}(x_1,\ x_2 + 1). \end{aligned}$$

Therefore (A.3) holds true by Lemma 2 (Zhuang and Li, 2010).

(b) $T_2v_{t+1}(x_1,x_2)-T_2v_{t+1}(x_1+1,x_2)\leqslant T_2v_{t+1}(x_1,x_2+1)-T_2v_{t+1}(x_1+1,x_2+1)$, ie,

$$\begin{aligned}&\max\{v_{t+1}(x_1,\ x_2+1)+r_2,\ v_{t+1}(x_1,\ x_2)\}\\&\quad-\max\{v_{t+1}(x_1+1,\ x_2+1)+r_2,\ v_{t+1}(x_1+1,\ x_2)\}\\&\quad\leqslant\max\{v_{t+1}(x_1,\ x_2+2)+r_2,\ v_{t+1}(x_1,\ x_2+1)\}\\&\quad-\max\{v_{t+1}(x_1+1,\ x_2+2)+r_2,\ v_{t+1}(x_1+1,\ x_2+1)\}.\end{aligned}\tag{A.4}$$

Since

$$\begin{aligned}&v_{t+1}(x_1,\ x_2+1)+r_2-(v_{t+1}(x_1+1,\ x_2+1)+r_2)\\&\quad\leqslant v_{t+1}(x_1,\ x_2+2)+r_2-(v_{t+1}(x_1+1,\ x_2+2)+r_2),\\&v_{t+1}(x_1,\ x_2+1)+r_2-(v_{t+1}(x_1+1,\ x_2+1)+r_2)\\&\quad\leqslant(=)v_{t+1}(x_1,\ x_2+1)-v_{t+1}(x_1+1,\ x_2+1),\\&v_{t+1}(x_1,\ x_2)-v_{t+1}(x_1+1,\ x_2)\\&\quad\leqslant v_{t+1}(x_1,\ x_2+2)+r_2-(v_{t+1}(x_1+1,\ x_2+2)+r_2),\\&v_{t+1}(x_1,\ x_2)-v_{t+1}(x_1+1,\ x_2)\\&\quad\leqslant v_{t+1}(x_1,\ x_2+1)-v_{t+1}(x_1+1,\ x_2+1),\end{aligned}$$

where the inequalities follow from submodularity of $v_{t+1}(x_1,x_2)$ that

$$\begin{aligned}&\Delta_1v_{t+1}(x_1,\ x_2+1)\leqslant\Delta_1v_{t+1}(x_1,\ x_2+2),\\&\Delta_1v_{t+1}(x_1,\ x_2)\leqslant\Delta_1v_{t+1}(x_1,\ x_2+2),\\&\Delta_1v_{t+1}(x_1,\ x_2)\leqslant\Delta_1v_{t+1}(x_1,\ x_2+1).\end{aligned}$$

hence (A.4) holds true. Therefore submodularity holds for $v_t(x_1,x_2)$, ie, $\Delta_1v_t(x_1,x_2)\leqslant\Delta_1v_t(x_1,x_2+1)$. □

Proof of Theorem 2 The structure of the optimal control policy presented in Theorem 2 follows directly from the submodularity of $v_t(x_1,x_2)$, which is proved in Lemma 2. □

Proof of Lemma 3 (a) First we show that $\Delta\tilde{v}_{T+1}(x)\leqslant\Delta\tilde{v}_{T+1}(x+1)$, ie,

$$\mathbf{E}[h(Y_{x+1})]-\mathbf{E}[h(Y_x)]\leqslant\mathbf{E}[h(Y_{x+2})]-\mathbf{E}[h(Y_{x+1})].\tag{A.5}$$

Note that (A.5) is equivalent to

$$h((x+1)A)-h(xA)\leqslant h((x+2)A)-h((x+1)A),$$

which is true since $h(\cdot)$ is convex. Hence, $\Delta\tilde{v}_{T+1}(x)\leqslant\Delta\tilde{v}_{T+1}(x+1)$ holds. Assume that $\Delta\tilde{v}_{t+1}(x)\leqslant\Delta\tilde{v}_{t+1}(x+1)$ and we need to show that $\Delta\tilde{v}_t(x)\leqslant\Delta\tilde{v}_t(x+1)$. The key is to show that $\Delta T_i\tilde{v}_{t+1}(x)\leqslant\Delta T_i\tilde{v}_{t+1}(x+1)$, where $T_i\tilde{v}_{t+1}(x)=\max\{\tilde{v}_{t+1}(x+1)+r_i,\ \tilde{v}_{t+1}(x)\}$. That is,

$$\begin{aligned}&\max\{\tilde{v}_{t+1}(x+1)+r_i,\ \tilde{v}_{t+1}(x)\}-\max\{\tilde{v}_{t+1}(x+2)\\&\quad+r_i,\ \tilde{v}_{t+1}(x+1)\}\leqslant\max\{\tilde{v}_{t+1}(x+2)+r_i,\ \tilde{v}_{t+1}(x+1)\}\\&\quad-\max\{\tilde{v}_{t+1}(x+3)+r_i,\ \tilde{v}_{t+1}(x+2)\},\end{aligned}\tag{A.6}$$

which follows by

$$\begin{aligned}&\tilde{v}_{t+1}(x+1)+r_i-(\tilde{v}_{t+1}(x+2)+r_i)\\&\quad\leqslant\tilde{v}_{t+1}(x+2)+r_i-(\tilde{v}_{t+1}(x+3)+r_i),\\&\tilde{v}_{t+1}(x+1)+r_i-\tilde{v}_{t+1}(x+1)\\&\quad\leqslant(=)\tilde{v}_{t+1}(x+2)+r_i-\tilde{v}_{t+1}(x+2),\\&\tilde{v}_{t+1}(x)-\tilde{v}_{t+1}(x+1)\\&\quad\leqslant\tilde{v}_{t+1}(x+2)+r_i-(\tilde{v}_{t+1}(x+3)+r_2),\\&\tilde{v}_{t+1}(x)-\tilde{v}_{t+1}(x+1)\\&\quad\leqslant\tilde{v}_{t+1}(x+1)-\tilde{v}_{t+1}(x+2).\end{aligned}$$

Therefore $\Delta\tilde{v}_t(x)\leqslant\Delta\tilde{v}_t(x+1)$ holds by Lemma 2 (Zhuang and Li, 2010).

(b) Note that

$$\begin{aligned}\Delta\tilde{v}(x)=\Delta\tilde{v}_{t+1}(x)+\sum_{i=1}^{n}\lambda_i[&\max\{r_i-\Delta\tilde{v}_{t+1}(x),\ 0\}\\&-\max\{r_i-\Delta\tilde{v}_{t+1}(x+1),\ 0\}\geqslant\Delta\tilde{v}_{t+1}(x),\end{aligned}$$

where the last inequality follows by $\Delta\tilde{v}_{t+1}(x)\leqslant\Delta\tilde{v}_{t+1}(x+1)$. □

Proof of Theorem 3 (a) Given that $r_i\geqslant r_j$. By $\Delta\tilde{v}_{t+1}(S^*_{it})>r_i\geqslant r_j$, it follows that $S^*_{it}\geqslant S^*_{jt}$.

(b) By Lemma 3(b), we have $\Delta\tilde{v}_t(S^*_{it})\geqslant\Delta\tilde{v}_{t+1}(S^*_{it})>r_i$ and it follows that $S^*_{it}\geqslant S^*_{i,t-1}$. □

Proof of Lemma 4 The proof follows similar arguments as that of Lemma 3 and thus is omitted. □

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