M&SOM MANUFACTURING & SERVICE OPERATIONS MANAGEMENT
Vol. 14, No. 3, Summer 2012, pp. 355–370
ISSN 1523-4614 (print) | ISSN 1526-5498 (online)

http://dx.doi.org/10.1287/msom.1120.0376

Markdown Pricing with Unknown Fraction of Strategic Customers

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A growing segment of the revenue management and pricing literature assumes "strategic" customers who are forward-looking in their pursuit of utility. Recognizing that such behavior may not be directly observable by a seller, we examine the implications of seller uncertainty over strategic customer behavior in a markdown pricing setting. We assume that some proportion of customers purchase impulsively in the first period if the price is below their willingness to pay, while other customers strategically wait for lower prices in the second period. We consider a two-period selling season in which the seller knows the aggregate demand curve but not the proportion of customers behaving strategically. We show that a robust pricing policy that requires no knowledge of the extent of strategic behavior performs remarkably well. We extend our model to a setting with stochastic demand, and show that the robust pricing policy continues to perform well, particularly as capacity is loosened or the problem is scaled up. Our results underscore the need to recognize strategic behavior, but also suggest that in many cases effective performance is possible without precise knowledge of strategic behavior.

Key words: consumer behavior; pricing and revenue management; retailing
History: Received: November 2, 2010; accepted: November 30, 2011. Published online in *Articles in Advance* May 4, 2012.

1. Introduction

Historically, much of the research in revenue management (RM) and dynamic pricing has assumed that customers are myopic and passive participants in the market, implying that a demand function describing how much customers are willing to pay serves as a sufficient summary of market response. A widening body of literature, however, considers "strategic" customers who pursue utility across time periods or products. Such customers may, for example, forgo immediate gratification or certain availability now in exchange for the possibility of future bargains. Strategic customer behavior complicates the seller's pricing problem. With time explicitly incorporated as another dimension of market response, a seller must not only know *how much* customers are willing to pay, but also *when* customers are willing to purchase, even as the timing of customer purchases is impacted by the seller's pricing decisions. As a result, demand models become more nuanced, and a single known demand curve no longer suits as a model of market response.

Despite the surge of interest in modeling strategic customer behavior, nearly all of the modeling work to date involving strategic behavior assumes that the parameters governing customer behavior are fully specified. Nevertheless, customers may be heterogeneous in their strategic behavior, individuals will not typically identify themselves as strategic, and strategic behavior may vary over time or products. We are aware of little work on estimating strategic behavior in the operations management literature, other than the work of Osadchiy and Bendoly (2011) in a laboratory setting. As Dubé et al. (2011) point out, estimating and predicting strategic behavior using historical sales data is difficult and subject to under-identification. For these reasons, sellers often operate unaware of strategic customer behavior or uncertain of its prevalence. An optimization model that ignores this uncertainty and its estimation may perform poorly. This paper seeks to answer important research questions that arise with this unknown. How costly is uncertainty around strategic behavior or erroneous assumptions about strategic behavior? Can a firm price effectively given this unknown?

We address these questions in a markdown pricing setting by extending a model of Zhang and Cooper (2008). A seller practicing markdown pricing successively reduces the price of a product over the course of a selling season. The classical justification for this practice is that it accomplishes indirect price discrimination over time. High valuation customers purchase at high prices early in the season, while low valuation customers purchase at "sale" prices later in the season (Kalish 1983). It is known that strategic customer behavior can erode the benefits of markdown pricing for the seller because high-valuation, forward-looking customers may opt to wait for lower prices at the end of the season (see Phillips 2005, Chap. 10).

We consider a two-period selling season in which a single product is first offered at a price p_1 in the regular period and then at a preannounced price $p_2 \le p_1$ in the clearance period. The seller has a capacity of c units of the product. Customers are heterogeneous in their behavior: A proportion α of them behave myopically, purchasing in the first period in which their willingness to pay (WTP) exceeds the prevailing price, while a proportion $1-\alpha$ of them behave strategically, optimizing the timing of their purchases based on the prices (p_1, p_2) and on anticipated availability in the clearance period. The seller knows the breakdown of customers' WTP, but does not know α.

For a deterministic, linear demand model, we derive a "robust" preannounced pricing policy that does not depend on the true α and that achieves revenues guaranteed to be no more than 2.05% worse than full information revenues. This compares with an 11% shortfall under a natural benchmark policy, namely when the seller ignores strategic behavior altogether. We explore the robustness of the policy to *customers'* assumptions of α (which can impact customers' behavior by altering their expectations of capacity rationing) and to a model involving stochastic demand. We provide evidence that the policy behaves capably over a wide range of settings that we characterize.

Our analysis has several important implications:

- While it is important to account for strategic customer behavior—ignoring it can eliminate the benefits of dynamic pricing—we find that the seller in our model can price remarkably well without specific knowledge of the prevalence of strategic behavior. This is a valuable insight given the difficulty in estimating strategic behavior and the fact that it may vary unpredictably over time.
- Good estimates of strategic customer behavior are not uniformly important across problem instances. For example, we find in our setup that understanding customer behavior becomes relatively unimportant as capacity becomes tight. In our stochastic setting, we find that our simple robust policy is effective except when load factors are moderate and the problem scale is small.
- Robust optimization can be used judiciously to simplify a complex problem under uncertainty. Our robust policy can be expressed in a closed form that does not require certain hard-to-estimate parameters.

The remainder of this paper is organized as follows. We review related streams of literature in §2. Section 3 formulates and analyzes our base deterministic model under full information. Section 4 analyzes a robust pricing policy assuming uncertainty about the customer behavior parameter. Section 5 considers a version of the model with stochastic demand. We conclude in §6 with a discussion of implications and open research directions. Proofs are included in the appendices.

2. Summary of Relevant Literature

Textbook treatments of RM and pricing include Talluri and van Ryzin (2004) and Phillips (2005). Bitran and Caldentey (2003), Elmaghraby and Keskinocak (2003), and Aviv and Vulcano (2012) review the literature on dynamic pricing. Our interest in the current paper is specifically on markdown pricing, on which Lazear (1986) is an early treatment. A number of authors have worked on markdown management with finite capacities (see Feng and Gallego 1995, Bitran and Mondschein 1997, Gallego and van Ryzin 1994, and other papers cited below).

Central to our model is the notion of strategic customer behavior, which has attracted considerable attention in the operations management literature in recent years, following the early seminal work of Stokey (1979) and Besanko and Winston (1990). Shen and Su (2007) survey recent streams of thought on customer behavior modeling for RM and auctions. Much recent work involving strategic customer behavior has centered on the game theoretic interactions between the seller and customers and, in some cases, competing sellers (e.g., Liu and van Ryzin 2008; Su 2007; Yin et al. 2009; Su and Zhang 2008, 2009). In early work, especially in marketing and economics, customer valuations and customer discounting have driven the timing of purchase decisions. Following Liu and van Ryzin (2008), a common driver of early purchases in the recent literature is customer expectations of inventory shortages. Various model features have been examined in recent work, including reservations mechanisms (Osadchiy and Vulcano 2010), quick response (Cachon and Swinney 2009), and customer visit costs (Cachon and Feldman 2010).

Our work assumes that customers are heterogeneous in their strategic behavior. In most previous works, customers are either all strategic or all myopic. Exceptions include Wilson et al. (2006), Su (2007), Lai et al. (2010), and Cachon and Swinney (2009), where typically a fraction of customers are strategic as in our model, but where the split of the population among strategic and myopic is known. Levin et al. (2009a, 2010) provide computational approaches to price optimization that allow for various degrees of strategic behavior. Ovchinnikov and Milner (2012) model strategic behavior as dependent on the seller's past decisions. They formulate a multiperiod dynamic programming model where the decisions are inventory allocations. In contrast to their work, we consider pricing decisions rather than allocation decisions, and our focus is on parameter uncertainty rather than the impact of decisions on future behavior.

Most of the papers on strategic customer behavior assume that the parameters of market response are known to the seller. Exceptions include simple sensitivity analyses (e.g., Besanko and Winston 1990, Cachon and Swinney 2009, Zhang and Cooper 2008, Anderson and Wilson 2003, Wilson et al. 2006) in which the authors examine the implications of the seller ignoring strategic customer behavior altogether. Strategic behavior is random in Ovchinnikov and Milner (2012), but is known in distribution. A few papers involving strategic behavior incorporate learning by customers rather than sellers (e.g., Liu and van Ryzin 2011, Gallego et al. 2008, Swinney 2011). A paper worth special mention is Levin et al. (2009b), in which a seller learns about a demand process that includes strategic customer behavior. Our work is distinguished from theirs in its analytical (versus algorithmic) focus and in its interest in strategic customer behavior as a specific source of uncertainty.

Decision making under uncertainty is central to our work, and we note that a few general approaches to handling uncertainty have appeared in the recent RM literature. A stream of recent research uses robust optimization techniques to make RM decisions in the face of unknowns. See Perakis and Roels (2010), Lan et al. (2010), Ball and Queyranne (2009), and Birbil et al. (2009). Our minimax approach in §4 is inspired by this idea, though we point out that these aforementioned papers do not consider strategic customer behavior.

As mentioned in §1, a justification for taking a robust approach to customer behavior in our context is the difficulty in estimating models with forward-looking consumers. There has been some work in economics and marketing on estimating dynamic discrete choice models for durable goods, for which forward-looking behavior of the kind we consider is plausible. It is clear that this remains a challenging question. Dubé et al. (2011) demonstrate the estimation of such models using conjoint survey data. Studies using historical sales data to estimate such models face an under-identification problem and often have to assume, for example, a value for customer discount factors (e.g., Song and Chintagunta 2003, Nair 2007). Chevalier and Goolsbee (2009) estimate discount factors, but they make use of assumptions specific to the college textbook industry. In a fashion retailing setting that is closer to our problem, Soysal (2007) highlights the importance of capturing strategic behavior in estimation. She divides customers into fashion-sensitive and a less fashion-sensitive, bargain-hunter segment, corresponding roughly to our myopic and strategic segments.

Finally, Zhang and Cooper (2008) study the pricing and rationing decision in a model that can be viewed as a special case of the deterministic model we motivate in §3. We use some of their results; specifically, our Proposition 2 echoes one of their results. However, our paper extends theirs significantly by modeling uncertainty with respect to α and by formulating and exploring the stochastic case.

3. A Deterministic Model with Full Information

Here we formulate and analyze a base model with deterministic demand. For expositional clarity, we first formulate our problem assuming the seller has full information. The solution to the base model will be useful later when we relax the full information assumption. Our assumption of deterministic demand is helpful for tractability and helps us to shine our spotlight on α, the uncertainty in which we are primarily interested. We will consider a natural extension with stochastic demand in §5.

Consider a seller who has c units of a product which he first offers at price p_1 in the "regular" period and then at price $p_2 \le p_1$ in the "clearance" period. The number of customers with valuations exceeding p is described by a linear demand curve $d(p) = (a - bp)^+$. This implies that customers' WTP is uniformly distributed on $[0, a/b]$. We assume that all customers arrive in the market in the regular period. This is a common assumption that is consistent, for example, with a short clearance period.

The assumption of preannounced pricing is common in the literature, dating back to the seminal paper of Stokey (1979). Preannounced pricing is natural with full information and deterministic demand because the seller expects to glean no information from early sales. This assumption is not completely innocuous when we introduce parameter uncertainty or stochastic demand. Our bounds on the deterministic problem in §4.2 suggest that contingent pricing policies bring only a modest revenue increase over preannounced pricing policies. In the stochastic case, analytical contingent pricing policies are unlikely for our problem even under full information. Aviv and Pazgal (2008), studying a distinct but related model to ours under full information, present an iterative numerical approach for optimizing contingent prices, but are unable to prove convergence.

As described in §1, each customer is one of two types: "myopic" or "strategic." A myopic customer purchases as soon as the price falls below her valuation. A strategic customer, on the other hand, times her purchase to maximize her surplus based on observed prices and anticipated availability in the clearance period. Customers are randomly split between myopic and strategic with splitting probabilities α and $1-\alpha$, respectively. If there is insufficient remaining capacity to satisfy clearance period demand, the seller will randomly allocate capacity to interested customers. The seller does not withhold or destroy capacity.

Our modeling of heterogeneous strategic behavior using two customer types, and our interpretations of "myopic" and "strategic," reflect the models of Wilson et al. (2006), Lai et al. (2010), and Cachon and Swinney (2009). We do not explicitly model the underlying source of this heterogeneity, but it is worth discussing why a customer would choose to purchase in the regular period versus the clearance period. As we reviewed in §2, the literature suggests two potential reasons: time discounting and sensitivity to availability. Our distinction between "myopic" and "strategic" customers is consistent with discounting. That is, myopic customers place no value on future consumption when making their decision in the regular period, while strategic customers do not discount future consumption. When capacity is finite, availability may also have an impact on the timing of customer purchases. That is, a strategic customer may choose to purchase in the regular period because of a risk of unavailability in the clearance period. When this risk is present, we can interpret "strategic" customers as those willing or able to consider risking clearance period unavailability in exchange for a lower price.

Uncertainty around α can significantly complicate the model, as the seller's revenues depend not only on the seller's beliefs, but also on the customers' beliefs about α. We make the simplifying assumption that customers share a common estimate $\hat{\alpha}_c$ of α. For the purposes of this section, we assume that the seller knows both α and $\hat{\alpha}_c$, but we will eventually relax this assumption.

Suppose the prices chosen by the seller are (p_1, p_2). A strategic customer's decision to purchase in the first period or wait until the second period depends on the expected fill rate θ in the clearance period given prices (p_1, p_2) and her estimate $\hat{\alpha}_c$. A strategic customer with valuation v will attempt to purchase in the regular period if $v-p_1 \geq \theta(v-p_2)$, which is equivalent to $v \geq (p_1-p_2\theta)/(1-\theta)$ assuming $\theta<1$.

Given $\hat{\alpha}_c$, a strategic customer can estimate the total number of strategic customers desiring to purchase in the regular period as $(1-\hat{\alpha}_c)[a-(b(p_1-p_2\theta))/(1-\theta)]^+$. The customers' estimated demands in the regular and clearance periods are, respectively,

$$\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta) = \hat{\alpha}_c(a-bp_1) + (1-\hat{\alpha}_c)\left(a-\frac{b(p_1-p_2\theta)}{1-\theta}\right)^+, \quad (1)$$

$$\lambda_2(p_1, p_2, \hat{\alpha}_c, \theta) = a-bp_2-\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta).$$

The fill rate in the clearance period anticipated by a strategic customer is therefore

$$f(p_1, p_2, \hat{\alpha}_c, \theta) = \min\left\{1, \frac{(c-\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta))^+}{\lambda_2(p_1, p_2, \hat{\alpha}_c, \theta)}\right\}. \quad (2)$$

In equilibrium, a strategic customer predicts the value of θ, which will satisfy the equation

$$\theta = f(p_1, p_2, \hat{\alpha}_c, \theta). \quad (3)$$

Observe that, given the customers' estimate $\hat{\alpha}_c$ and conditional on the prices p_1 and p_2, the predicted fill rate does not depend on α.

The seller's objective is to maximize revenue subject to constraint (3). We can write the seller's optimization problem as

$$\begin{aligned} \max_{a/b\geq p_1\geq p_2\geq 0,\, \theta} \quad & p_1\min\{c, \lambda_1(p_1, p_2, \alpha, \theta)\} \\ & + p_2\min\{(c-\lambda_1(p_1, p_2, \alpha, \theta))^+, \\ & \qquad \lambda_2(p_1, p_2, \alpha, \theta)\} \quad (4) \\ \text{s.t.} \quad & \theta = \min\left\{1, \frac{(c-\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta))^+}{\lambda_2(p_1, p_2, \hat{\alpha}_c, \theta)}\right\}. \end{aligned}$$

This formulation assumes that in the event there are multiple solutions to (3), the one yielding the highest revenue will be selected.

Let

$$\Delta(\alpha, \hat{\alpha}_c) = \left(\sqrt{1-\alpha+\hat{\alpha}_c}-\sqrt{1-\alpha}\right)^2,$$

$$\Gamma(\alpha, \hat{\alpha}_c) = \frac{(1-2\alpha+2\hat{\alpha}_c)\sqrt{\Delta(\alpha, \hat{\alpha}_c)}}{1-\alpha+\hat{\alpha}_c}-\hat{\alpha}_c.$$

The following proposition provides the optimal solution to problem (4) under full information. We prove it in Appendix B.

Proposition 1. *An optimal solution to* (4) *is given by*

$$\begin{aligned} &\left(p_1^*(\alpha, \hat{\alpha}_c, c), p_2^*(\alpha, \hat{\alpha}_c, c), \theta^*(\alpha, \hat{\alpha}_c, c)\right) \\ &= \begin{cases} \left(\dfrac{(3-\alpha)a}{(4-\alpha)b}, \dfrac{(2-\alpha)a}{(4-\alpha)b}, 1\right) \\ \quad \text{if } c \geq \dfrac{2a}{4-\alpha},\ [2a-c(4-\alpha)]^2 \\ \qquad + c^2(4-\alpha)(\alpha-\Delta(\alpha, \hat{\alpha}_c)) \geq 0, \\ \left(\dfrac{2a-[2-\Delta(\alpha, \hat{\alpha}_c)]c}{2b}, \dfrac{2a-[2-\Gamma(\alpha, \hat{\alpha}_c)]c}{2b}, \theta^\dagger\right) \\ \quad \text{if } c < \dfrac{2a}{4-\alpha},\ \alpha < \Delta(\alpha, \hat{\alpha}_c),\ \text{or } c \geq \dfrac{2a}{4-\alpha}, \\ \qquad [2a-c(4-\alpha)]^2 + c^2(4-\alpha)(\alpha-\Delta(\alpha, \hat{\alpha}_c)) < 0, \\ \left(\dfrac{2a-c}{2b}, \dfrac{a-c}{b}, 1\right) \quad \text{if } c < \dfrac{2a}{4-\alpha},\ \alpha \geq \Delta(\alpha, \hat{\alpha}_c), \end{cases} \end{aligned} \quad (5)$$

where $\theta^\dagger = (1-\alpha+\hat{\alpha}_c - \sqrt{(1-\alpha)(1-\alpha+\hat{\alpha}_c)})/\hat{\alpha}_c$. *The corresponding revenue is given by*

$$v^*(\alpha,\hat{\alpha}_c,c) = \begin{cases} \dfrac{a^2}{b(4-\alpha)} & \text{if } c \ge \dfrac{2a}{4-\alpha}, \\ & [2a-c(4-\alpha)]^2 + c^2(4-\alpha)(\alpha-\Delta(\alpha,\hat{\alpha}_c)) \ge 0, \\ \dfrac{c(4a-[4-\Delta(\alpha,\hat{\alpha}_c)]c)}{4b} & \\ & \text{if } c < \dfrac{2a}{4-\alpha}, \alpha < \Delta(\alpha,\hat{\alpha}_c), \text{ or } c \ge \dfrac{2a}{4-\alpha}, \\ & [2a-c(4-\alpha)]^2 + c^2(4-\alpha)(\alpha-\Delta(\alpha,\hat{\alpha}_c)) < 0, \\ \dfrac{(4a-(4-\alpha)c)c}{4b} & \text{if } c < \dfrac{2a}{4-\alpha}, \alpha \ge \Delta(\alpha,\hat{\alpha}_c). \end{cases} \tag{6}$$

Although this solution is closed-form, it is fairly complex and difficult to analyze when we move to the case with incomplete information in the following sections. The solution, however, can be considerably simplified in a special case where both the seller and customers know α. This special case was analyzed in Zhang and Cooper (2008). We reproduce their result below.

Proposition 2 (Zhang and Cooper 2008). *Assuming* $\hat{\alpha}_c = \alpha$, *the deterministic problem* (4) *is solved by* $\theta = 1$ *and*

$$(p_1^*(\alpha,c), p_2^*(\alpha,c)) = \begin{cases} \left(\dfrac{(3-\alpha)a}{(4-\alpha)b}, \dfrac{(2-\alpha)a}{(4-\alpha)b}\right) & \text{if } c \ge \dfrac{2a}{4-\alpha}, \\ \left(\dfrac{2a-c}{2b}, \dfrac{a-c}{b}\right) & \text{if } c < \dfrac{2a}{4-\alpha}, \end{cases} \tag{7}$$

with corresponding revenue

$$v^*(\alpha,c) = \begin{cases} \dfrac{a^2}{b(4-\alpha)} & \text{if } c \ge \dfrac{2a}{4-\alpha}, \\ \dfrac{(4a-(4-\alpha)c)c}{4b} & \text{if } c < \dfrac{2a}{4-\alpha}. \end{cases} \tag{8}$$

It is straightforward to show that the solution given in Proposition 2 follows as a special case of Proposition 1 if we substitute $\hat{\alpha}_c = \alpha$ into the solution given in Proposition 1.

The prices specified by Proposition 2 satisfy $d(p_2^*) \le c$, implying that there is no rationing (i.e., $\theta = 1$) in equilibrium. With no rationing risk, strategic customers will never purchase in the regular period, and will purchase in the clearance period if their valuation exceeds p_2. We emphasize that our model as formulated in Equation (4) *allows* for rationing, but that it is optimal for the seller to price such that demand is no larger than capacity. We put this in context by observing that both pricing and rationing are potential levers for managing consumer demand, and the value of rationing tends to be larger in the absence of pricing flexibility (see, e.g., Liu and van Ryzin 2008). As we see in Proposition 1 (specifically, the second case in Equation (5)), the no-rationing result will not necessarily hold up when we allow for arbitrary $\hat{\alpha}_c$. In addition, §5 will show that there will generally be a positive probability of rationing in the stochastic version of the model.

We make two further observations about Proposition 2 that will bear on our results in the following section. First, as long as capacity is sufficiently loose, the optimal prices become larger as strategic behavior becomes more prevalent (i.e., as α gets smaller). Second, tight capacity leads to an increase in prices as intuition would suggest. Interestingly, when strategic customer behavior and limited capacity are overlaid, only one of them impacts prices for a given problem instance. That is, α does not figure into the optimal prices when capacity is sufficiently tight, while c does not impact prices when capacity is sufficiently loose. This observation foreshadows a finding to come, i.e., that uncertainty around α is irrelevant for sufficiently tight capacity.

4. Pricing When α Is Unknown

Section 3 considered an idealized situation in which the seller knows α and $\hat{\alpha}_c$. Here we consider cases where the seller does not know α. We derive both an appealing policy and bounds on the value of knowing α.

4.1. The Case $\hat{\alpha}_c = \hat{\alpha}_s$

Given the seller operates without knowledge of α, we solve in closed form the problem of minimizing the seller's maximum shortfall compared with an optimal full-information policy. We restrict our attention to a class of policies whereby the seller assumes an estimate $\hat{\alpha}_s$ of α and sets prices using Equation (7) as if $\alpha = \hat{\alpha}_s$. We make three remarks about this choice.

1. Even when the seller assumes $\hat{\alpha}_s$, the seller's uncertainty about the consumer belief $\hat{\alpha}_c$ is not resolved. Obviously, the customers' estimate $\hat{\alpha}_c$ can play a role in our analysis. Throughout this subsection we assume that $\hat{\alpha}_c = \hat{\alpha}_s$, i.e., customers use the same estimate the seller uses to set prices (and this fact is known to the seller). This is especially reasonable when the customers can reverse-engineer $\hat{\alpha}_s$ from the seller's prices. (In our model, we can solve for $\hat{\alpha}_s$ as a function of prices as long as $c \ge 2a/(4-\hat{\alpha}_s)$.) We explore the robustness of the resulting prices under alternate assumptions in §4.2.

2. Preannounced prices are natural within this class of policies. A seller assuming $\alpha = \hat{\alpha}_s$ and facing deterministic demand expects to glean no information

from the regular period and therefore will be willing to commit to both p_1 and p_2 at the beginning of the season.

3. We will identify a policy in this class with a 2.05% performance guarantee relative to the full-information optimal policy, suggesting that restricting ourselves to this class of policies comes at a low cost.

If the seller had a belief about α, in the form of a probability distribution around α, one approach would be to formulate and solve the problem of maximizing the seller's expected revenues with respect to its belief. We do not pursue such an approach for two reasons. First, it is not obvious how a seller would derive such a belief. Second, depending on what we assume about the customers' beliefs, such an approach does not lend itself to insightful solutions. Instead, we focus on a "robust" approach that accounts for the fact that α is unknown. We will show that a relatively simple, closed-form robust policy performs quite well, and we explicitly characterize a strong performance bound.

The total demand for the season is given by

$$d(p_2^*(\hat{\alpha}_s, c)) = a - bp_2^*(\hat{\alpha}_s, c) = \begin{cases} c & \text{if } c < \dfrac{2a}{4-\hat{\alpha}_s}, \\[2ex] \dfrac{2a}{4-\hat{\alpha}_s} & \text{if } c \geq \dfrac{2a}{4-\hat{\alpha}_s}. \end{cases}$$

Hence $d(p_2^*(\hat{\alpha}_s, c)) \leq c$ as mentioned earlier. The corresponding revenue is

$$\begin{aligned} \hat{v}(\alpha, \hat{\alpha}_s, c) &= \alpha d(p_1^*(\hat{\alpha}_s, c)) p_1^*(\hat{\alpha}_s, c) \\ &\quad + [d(p_2^*(\hat{\alpha}_s, c)) - \alpha d(p_1^*(\hat{\alpha}_s, c))] p_2^*(\hat{\alpha}_s, c) \\ &= \begin{cases} \dfrac{c(4a-(4-\alpha)c)}{4b} & \text{if } c < \dfrac{2a}{4-\hat{\alpha}_s}, \\[2ex] \dfrac{a^2(4-2\hat{\alpha}_s+\alpha)}{(4-\hat{\alpha}_s)^2 b} & \text{if } c \geq \dfrac{2a}{4-\hat{\alpha}_s}. \end{cases} \end{aligned}$$

It is straightforward to show that revenues generated under the assumption $\hat{\alpha}_s$ are lower than $v^*(\alpha, c)$, the optimal revenues when both seller and customers know α. That is, $\hat{v}(\alpha, \hat{\alpha}_s, c) \leq v^*(\alpha, c)$. We can measure the relative revenue shortfall because of the use of the estimated value $\hat{\alpha}_s$ relative to $v^*(\alpha, c)$ as

$$g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c) = \frac{v^*(\alpha, c) - \hat{v}(\alpha, \hat{\alpha}_s, c)}{v^*(\alpha, c)}. \tag{9}$$

We will sometimes refer to this quantity as "relative revenue shortfall" or simply "shortfall." Note that $g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c)$ will always take a value in $[0, 1]$ and can be interpreted as the percentage loss from assuming $\hat{\alpha}_s$ rather than the true α. A robust choice for $\hat{\alpha}_s$ from the perspective of relative revenue shortfall solves the minimax problem

$$\bar{g}^* = \min_{\hat{\alpha}_s \in [0,1]} \max_{\alpha \in [0,1]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c). \tag{10}$$

The formulation (10) minimizes the maximum relative revenue shortfall over all possible α values. The following proposition establishes the minimax solution for this problem.

Proposition 3. *Let $\hat{\alpha}_s^*(c)$ be the optimal solution to* (10). *Then*

$$\hat{\alpha}_s^*(c) = \begin{cases} \textit{any value in } [0,1] & \textit{if } c < \dfrac{a}{2}, \\[2ex] 2 - \dfrac{a^2}{2(3c-a)(a-c)} & \textit{if } \dfrac{a}{2} \leq c \leq \dfrac{2a}{3}, \\[2ex] \dfrac{1}{2} & \textit{if } c > \dfrac{2a}{3}, \end{cases} \tag{11}$$

with associated revenue

$$\begin{aligned} &\hat{v}(\alpha, \hat{\alpha}_s^*(c), c) \\ &\quad = \begin{cases} \dfrac{c(4a-(4-\alpha)c)}{4b} & \textit{if } c < \dfrac{a}{2}, \\[2ex] \dfrac{a^2(4-2\hat{\alpha}_s^*(c)+\alpha)}{(4-\hat{\alpha}_s^*(c))^2 b} & \textit{if } \dfrac{a}{2} \leq c \leq \dfrac{2a}{3}, \\[2ex] \dfrac{4a^2(3+\alpha)}{49b} & \textit{if } c > \dfrac{2a}{3}. \end{cases} \end{aligned} \tag{12}$$

The corresponding minimax relative revenue shortfall is given by

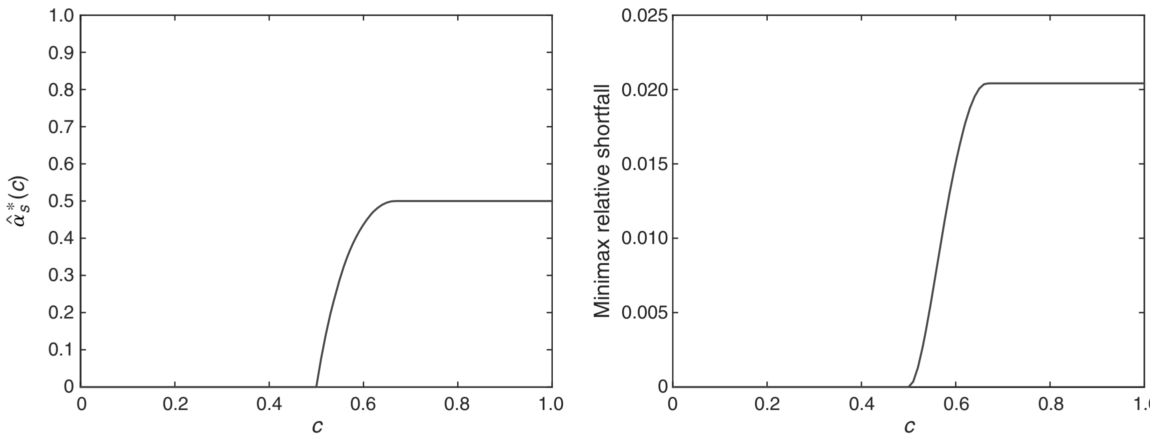
$$\begin{aligned} \bar{g}^* &= \max_{\alpha \in [0,1]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s^*(c), c) \\ &= \begin{cases} 0 & \textit{if } c < \dfrac{a}{2}, \\[2ex] \left(\dfrac{4(3c-a)(a-c)-a^2}{4(3c-a)(a-c)+a^2}\right)^2 & \textit{if } \dfrac{a}{2} \leq c \leq \dfrac{2a}{3}, \\[2ex] \dfrac{1}{49} & \textit{if } c > \dfrac{2a}{3}. \end{cases} \end{aligned} \tag{13}$$

Figure 1 is a visual representation of Proposition 3. The figure uses $a = 1$, though we note that the patterns are similar for other values of a.

We make a few observations about Proposition 3. First, we have $g^{\mathrm{R}}(\alpha, \hat{\alpha}_s^*(c), c) \leq 1/49 < 2.05\%$ for all a, b, and c. Thus, knowledge of the fraction of strategic customers is worth no more than about 2% of revenues achievable when both seller and customers know α. When capacity is very tight, knowledge of strategic behavior is inconsequential to setting prices. Recall from Equation (7) that prices are independent of α for tight capacity. When capacity is loose, a seller without knowledge of α can nearly achieve optimal full-information revenues simply by setting prices as if $\hat{\alpha}_s = 1/2$.

We contrast the finding of Proposition 3 with two naive but natural approaches, namely where the seller either prices as if all customers are myopic ($\hat{\alpha}_s = 1$) or as if all customers are strategic ($\hat{\alpha}_s = 0$). The former is a natural assumption for a seller who does not recognize the existence of strategic behavior or who

Figure 1 **Optimal Minimax Solution $\hat{\alpha}_s^*(c)$ (Left) and Resulting Minimax Relative Shortfall (Right) as Functions of the Capacity c**



Note. For $c < 1/2$, any choice for $\hat{\alpha}_s$ on [0,1] is optimal.

does not know how to incorporate strategic behavior into decision making. This seems to be a common assumption in practice and, as mentioned in §2, in much of the academic literature on dynamic pricing. The latter approach is assumed by much of the recent stream of literature on strategic customer behavior, with a few exceptions detailed in §2.

Making use of Equation (9), we evaluate these assumptions for various capacities c and true values of α. When the seller assumes all customers are myopic ($\hat{\alpha}_s = 1$),

$$g^{\mathrm{R}}(\alpha, 1, c) = \begin{cases} 0 & \text{if } c \le \dfrac{2a}{4-\alpha}, \\ 1 - \dfrac{c(4-\alpha)(4a-(4-\alpha)c)}{4a^2} & \text{if } \dfrac{2a}{4-\alpha} < c \le \dfrac{2a}{3}, \\ \dfrac{(1-\alpha)^2}{9} & \text{if } c > \dfrac{2a}{3}. \end{cases}$$

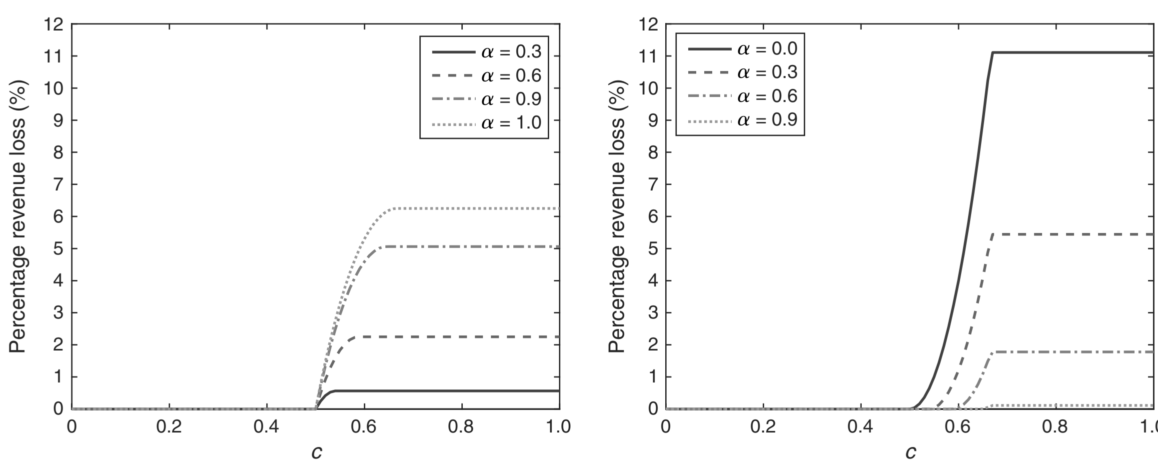
When the seller assumes all customers are strategic ($\hat{\alpha}_s = 0$),

$$g^{\mathrm{R}}(\alpha, 0, c) = \begin{cases} 0 & \text{if } c \le \dfrac{a}{2}, \\ 1 - \dfrac{a^2(4+\alpha)}{(4c(4a-(4-\alpha)c))} & \text{if } \dfrac{a}{2} < c \le \dfrac{2a}{4-\alpha}, \\ \dfrac{\alpha^2}{16} & \text{if } c > \dfrac{2a}{4-\alpha}. \end{cases}$$

Figure 2 depicts these results graphically, assuming $a = 1$.

These "all-or-nothing" assumptions perform the worst when c is large and, as we might expect, when $\hat{\alpha}_s$ is farthest from α. When strategic behavior is ignored ($\hat{\alpha}_s = 1$), $g^{\mathrm{R}}(\alpha, 1, c)$ reaches its maximum for large c and $\alpha = 0$, in which case we have $g^{\mathrm{R}}(0, 1, \infty) = 1/9 \approx 11.11\%$. This is practically significant and over five times as large as the worst-case optimality gap under the minimax choice of $\hat{\alpha}_s$.

Figure 2 **Percentage Revenue Loss Under Extreme Assumptions: Assuming $\hat{\alpha}_s = 0$ (Left); Assuming $\hat{\alpha}_s = 1$ (Right)**



Also, in the uncapacitated case, revenues are $w(\alpha, \hat{p}_1(1,\infty), \hat{p}_2(1,\infty)) = (2+\alpha)a^2/(9b)$. When $\alpha < 1/4$, this revenue is smaller than the optimal single-price revenue, which is easily shown to be $v^{\mathrm{SP}} = a^2/(4b)$. Thus, a seller who ignores strategic customer behavior when setting two prices might have been better off with a static pricing policy. On the other hand, when the seller assumes that all customers are strategic ($\hat{\alpha}_s = 0$), the relative shortfall is the largest when c is large and the true α is 1. In this case, $g^{\mathrm{R}}(1, 0, \infty) = 1/16 = 6.25\%$, a factor of three larger than the worst-case minimax performance.

4.2. The Case with General $\hat{\alpha}_c$

Our discussion in §4.1 focused on a case in which we assumed that customers matched their estimate of α with the seller's, i.e., $\hat{\alpha}_c = \hat{\alpha}_s$. In this subsection we present results for the case in which customers share an arbitrary belief $\hat{\alpha}_c$ that may differ from $\hat{\alpha}_s$ and is unknown to the seller.

A key observation we made about Proposition 2 is that customers face no rationing risk in equilibrium under optimal prices. Here we are considering the model whose full information solution appears in Proposition 1, in which capacity will be rationed under full-information optimal prices for some problem instances. Figure 3 indicates the instances for which rationing occurs, in terms of the parameters α and $\hat{\alpha}_c$.

Calculating a minimax policy based on the revenue function (6) is analytically challenging. We can, however, analytically characterize the relative shortfall of the simple robust policy given in Proposition 3, developed under the assumption that $\hat{\alpha}_c = \hat{\alpha}_s$. Let $\hat{v}(\alpha, \hat{\alpha}_s^*(c), c)$ be the revenue received under the simple robust policy; this revenue is given in Equation (12). A benefit of this policy is that it does not depend on $\hat{\alpha}_c$, a quantity that might be difficult for the seller to know because $d(p_2) \le c$ under this policy. The relative shortfall is given by

$$g^{\dagger}(\alpha, \hat{\alpha}_s^*(c), \hat{\alpha}_c, c) = \frac{v^*(\alpha, \hat{\alpha}_c, c) - \hat{v}(\alpha, \hat{\alpha}_s^*(c), c)}{v^*(\alpha, \hat{\alpha}_c, c)}, \quad (14)$$

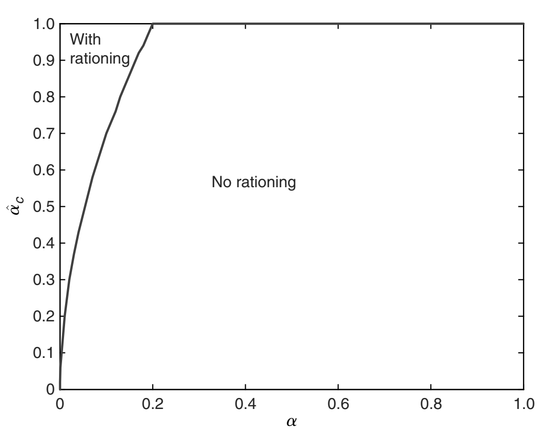
which differs from Equation (9) because we are measuring shortfall against the optimal full-information revenues given in Proposition 1 rather than in Proposition 2. Proposition 4 characterizes $g^{\dagger}(\alpha, \hat{\alpha}_s^*(c), \hat{\alpha}_c, c)$ by evaluating Equation (14) using the expressions in Equations (6) and (12). We omit the details for brevity.

Proposition 4. *The relative revenue shortfall $g^{\dagger}(\alpha, \hat{\alpha}_s^*(c), \hat{\alpha}_c, c)$ is given by the following equation.*

$$g^{\dagger}(\alpha, \hat{\alpha}_s^*(c), \hat{\alpha}_c, c) = \begin{cases} 0 & \text{if } c < \dfrac{a}{2},\ \alpha \ge \Delta(\alpha, \hat{\alpha}_c), \\[2ex] \dfrac{c[\Delta(\alpha, \hat{\alpha}_c) - \alpha]}{4a - (4 - \Delta(\alpha, \hat{\alpha}_c))c} & \text{if } c < \dfrac{a}{2},\ \alpha < \Delta(\alpha, \hat{\alpha}_c), \\[2ex] 1 - \dfrac{4a^2(4 - 2\hat{\alpha}_s^*(c) + \alpha)}{c(4 - \hat{\alpha}_s^*(c))^2[4a - (4-\alpha)c]} & \text{if } \dfrac{a}{2} \le c < \dfrac{2a}{4-\alpha},\ \alpha \ge \Delta(\alpha, \hat{\alpha}_c), \\[2ex] 1 - \dfrac{4a^2(4 - 2\hat{\alpha}_s^*(c) + \alpha)}{c(4 - \hat{\alpha}_s^*(c))^2(4a - [4 - \Delta(\alpha, \hat{\alpha}_c)]c)} & \text{if } \dfrac{a}{2} \le c < \dfrac{2a}{4-\alpha},\ \alpha < \Delta(\alpha, \hat{\alpha}_c), \text{ or } \\ & \quad \dfrac{2a}{4-\alpha} \le c \le \dfrac{2a}{3},\ [2a - c(4-\alpha)]^2 \\ & \quad + c^2(4-\alpha)(\alpha - \Delta(\alpha, \hat{\alpha}_c)) < 0, \\[2ex] \left(\dfrac{\alpha - \hat{\alpha}_s^*(c)}{4 - \hat{\alpha}_s^*(c)}\right)^2 & \text{if } \dfrac{2a}{4-\alpha} \le c \le \dfrac{2a}{3}, \\ & \quad [2a - c(4-\alpha)]^2 + c^2(4-\alpha)(\alpha - \Delta(\alpha, \hat{\alpha}_c)) \ge 0, \\[2ex] 1 - \dfrac{16a^2(3+\alpha)}{49c(4a - [4 - \Delta(\alpha, \hat{\alpha}_c)]c)} & \text{if } c > \dfrac{2a}{3}, \\ & \quad [2a - c(4-\alpha)]^2 + c^2(4-\alpha)(\alpha - \Delta(\alpha, \hat{\alpha}_c)) < 0, \\[2ex] \dfrac{(1-2\alpha)^2}{49} & \text{if } c > \dfrac{2a}{3},\ [2a - c(4-\alpha)]^2 \\ & \quad + c^2(4-\alpha)(\alpha - \Delta(\alpha, \hat{\alpha}_c)) \ge 0. \end{cases}$$

The simple robust policy performs quite well under the alternate assumption about $\hat{\alpha}_c$. For the majority of the parameter space (for example, when $\hat{\alpha}_c \le \hat{\alpha}_s$), we find our 2.05% shortfall bound to be robust. Proposition 4 implies that the relative shortfall of the robust policy is bounded by 4.62%, but shortfalls this large occur only in extreme cases in which α is very small and $\hat{\alpha}_c$ is very large. In these cases, the full-information revenue is enhanced because a seller with full-information can take advantage of customer

Figure 3 Regions With and Without Rationing in the Full Information Optimal Solution When $c < 2a/(4-\alpha)$



Note. The region with rationing is smaller for $c \ge 2a/(4-\alpha)$.

beliefs to induce customers to purchase in the regular period. The *average* shortfall, averaged over instances with α and $\hat{\alpha}_c$ running from 0 to 1 and c running from 0.01 to 1, is approximately 0.3%.

As mentioned earlier, our robust policy does not require the seller to know customer belief $\hat{\alpha}_c$. If we instead think of a seller who assumes a (possibly incorrect) value for α, assumes a (possibly incorrect) value for $\hat{\alpha}_c$, and computes optimal prices via Proposition 1 given these assumptions, such a seller can collect revenues 36.82% short of the revenues of a seller who makes the correct assumptions about α and $\hat{\alpha}_c$. The worst case occurs when the seller assumes $\alpha = 1$ and $\hat{\alpha}_c = 0$ even as the true values are $\alpha = 0$ and $\hat{\alpha}_c = 0$. Our shortfall bound of 4.62% compares quite well with this worst case.

We remark that our robust policy is based on the seller using a point estimate $\hat{\alpha}_s$ and adopting a preannounced pricing strategy. Other models might assume (i) the seller has a prior probability distribution on α, and/or (ii) the seller uses a contingent pricing strategy that potentially allows price p_2 to depend on sales in the regular period. If we assume that the customers' estimate of α is fixed at $\hat{\alpha}_c$, it is straightforward to show that $g^\dagger(\alpha, \hat{\alpha}_s^*, \hat{\alpha}_c)$ upper bounds the best possible *expected* revenue shortfall achievable within this more sophisticated class of policies. This suggests that our restriction to preannounced pricing policies comes at a low cost to the seller.

5. The Stochastic Case

In this section, we extend our base model to the case in which demand follows a Poisson distribution. Because the Poisson distribution has unbounded support, Poisson demand induces a positive probability of rationing in the clearance period under all nondegenerate problem parameters, and thus differs from the case in §4. Nevertheless, the deterministic problem remains a limiting case of the stochastic model in two regimes we explore. Accordingly, the robust policy in §4 performs quite well in the stochastic problem for a wide range of parameter settings. While we view the deterministic problem to be interesting in its own right, this section reveals that it can also be motivated as an approximation of the arguably more realistic stochastic problem.

Paralleling the organization of §§3 and 4, we first formulate the problem assuming full information. We then examine the revenue shortfall for a policy (specifically, the robust policy implied by Proposition 3 and Equation (7)) that does not assume knowledge of α. We naturally extend the model of §3 by making the number of customers with WTP exceeding p a Poisson random variable with mean $d(p) = (a - bp)^+$. As before, we assume that customers are split according to probabilities α and $1-\alpha$ into "myopic" and "strategic" types. We also assume as before that customers share an estimate $\hat{\alpha}_c$ of α.

Given $\hat{\alpha}_c$, a strategic customer estimates the total number of strategic customers desiring to purchase in the regular period as a Poisson random variable with mean $(1-\hat{\alpha}_c)[a - b(p_1 - p_2\theta)/(1-\theta)]^+$. The customers' estimated mean demands in the regular and clearance periods are $\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta)$ and $\lambda_2(p_1, p_2, \hat{\alpha}_c, \theta)$, respectively, where these quantities are defined in Equation (1).

We assume a strategic customer does not observe the arrivals of other customers and has only distributional information on the arrivals. Conditional on her own arrival, the expected fill rate in the clearance period anticipated by a strategic customer is therefore

$$f(p_1, p_2, \hat{\alpha}_c, \theta) = \mathsf{E}\left[\min\left\{1, \frac{(c - N[\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta)])^+}{N[\lambda_2(p_1, p_2, \hat{\alpha}_c, \theta)]}\right\} \;\middle|\; N[\lambda_2(p_1, p_2, \hat{\alpha}_c, \theta)] \geq 1\right], \quad (15)$$

where we use the notation $N[x]$ to denote a Poisson random variable with rate x.

Let $q_i(\lambda)$ be the probability that a Poisson random variable with mean λ takes the value i. Observe that (i) if demand in the regular period is greater or equal to c, the fill rate in the clearance period is 0, and (ii) if total demand is less than or equal to c, the fill rate in the clearance period is 1. We therefore have

$$\begin{aligned} f(p_1, p_2, \hat{\alpha}_c, \theta) &= \sum_{i=0}^{c-1}\sum_{j=1}^{c-i} \frac{q_i(\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta))q_j(\lambda_2(p_1, p_2, \hat{\alpha}_c, \theta))}{1 - q_0(\lambda_2(p_1, p_2, \hat{\alpha}_c, \theta))} \\ &\quad + \sum_{i=0}^{c-1}\sum_{j=c-i+1}^{\infty} \frac{(c-i)q_i(\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta))q_j(\lambda_2(p_1, p_2, \hat{\alpha}_c, \theta))}{j[1 - q_0(\lambda_2(p_1, p_2, \hat{\alpha}_c, \theta))]}. \end{aligned} \quad (16)$$

Next we turn to the seller. Assume as in §3 that the seller knows both α and the customer estimate $\hat{\alpha}_c$. Given p_1, p_2, α, and θ, the seller's calculated mean demands in the regular and clearance periods are given by $\lambda_1(p_1, p_2, \alpha, \theta)$ and $\lambda_2(p_1, p_2, \alpha, \theta)$, respectively.

Because the seller knows $\hat{\alpha}_c$, he can infer Equation (3) used by the customers to predict θ. Therefore, the seller's sales in the regular and clearance periods are given by the random variables

$$\begin{aligned} S_1(p_1, p_2, \alpha, \theta) &= \min\{c, N[\lambda_1(p_1, p_2, \alpha, \theta)]\}, \\ S_2(p_1, p_2, \alpha, \theta) &= \min\{c - S_1(p_1, p_2, \alpha, \theta), \\ &\qquad N[\lambda_2(p_1, p_2, \alpha, \theta)]\}. \end{aligned}$$

The corresponding expected revenue is

$$\phi(p_1, p_2, \alpha, \theta) = p_1 \mathsf{E}[S_1(p_1, p_2, \alpha, \theta)] + p_2 \mathsf{E}[S_2(p_1, p_2, \alpha, \theta)].$$

The seller's objective is to maximize expected revenue subject to the equilibrium constraint $\theta = f(p_1, p_2, \hat{\alpha}_c, \theta)$:

$$\begin{aligned} \pi(\alpha, \hat{\alpha}_c) = \max_{a/b \geq p_1 \geq p_2 \geq 0,\, \theta} \quad & \phi(p_1, p_2, \alpha, \theta) \\ \text{s.t.} \quad & \theta = f(p_1, p_2, \hat{\alpha}_c, \theta). \end{aligned} \tag{17}$$

As with formulation (4), here we assume that if there are multiple solutions to the equilibrium constraint, the one yielding the highest revenue will be selected.

Problem (17) is analytically challenging because prices impact demand in each period in a complex way through the fill rate θ. However, the deterministic problem is a good approximation in some settings. In Appendix A, we show that the stochastic problem is equivalent to the deterministic problem in two distinct asymptotic regimes: when we let $c \to \infty$; and when capacity equals nc, mean demand equals $n(a - bp)^+$, and we let $n \to \infty$. The first result is intuitive if we consider that in an uncapacitated problem $f(p_1, p_2, \hat{\alpha}_c, \theta) = 1$ regardless of the seller's policy; therefore, expected sales in each period becomes equal to expected demand. The second result relies on the Poisson random variables approaching their fluid approximations via a Strong Law of Large Numbers argument. Such arguments are common in RM analyses; see Osadchiy and Vulcano (2010) for a recent example.

It follows that we expect the deterministic problem to best approximate the stochastic problem when capacity is relatively loose and/or when capacity and customer volume are both large. This is intuitive at a high level if we notice that the key difference between the stochastic and deterministic problems is the difference between $\mathsf{E}[\min\{c - N[\lambda_1(p_1, p_2, \alpha, \theta)], N[\lambda_2(p_1, p_2, \alpha, \theta)]\}]$ and $\min\{c - \lambda_1(p_1, p_2, \alpha, \theta), \lambda_2(p_1, p_2, \alpha, \theta)\}$. Our robust policy works well as an approximation in the stochastic problem when the two expressions are nearly equal. The two expressions are trivially equal when demand is deterministic (which happens asymptotically as the problem is scaled up) and when $c \to \infty$.

For the sake of simplicity, we restrict ourselves, as in §4.1, to the case $\hat{\alpha}_c = \hat{\alpha}_s$, in which customers take the seller's estimate of α. Because the expectations on the right-hand side of Equation (15) resist analytical solution, we were unable to analytically solve problem (17) even under full information. Instead we numerically evaluate the performance of a particular policy, namely the minimax prices implied by Proposition 3 and Equation (7). We refer to this policy as the "fluid robust" or "robust" policy. We also numerically evaluate the revenues under an optimal full-information policy in which seller and customers know α, enabling us to compute the relative revenue shortfall of our pricing policy.

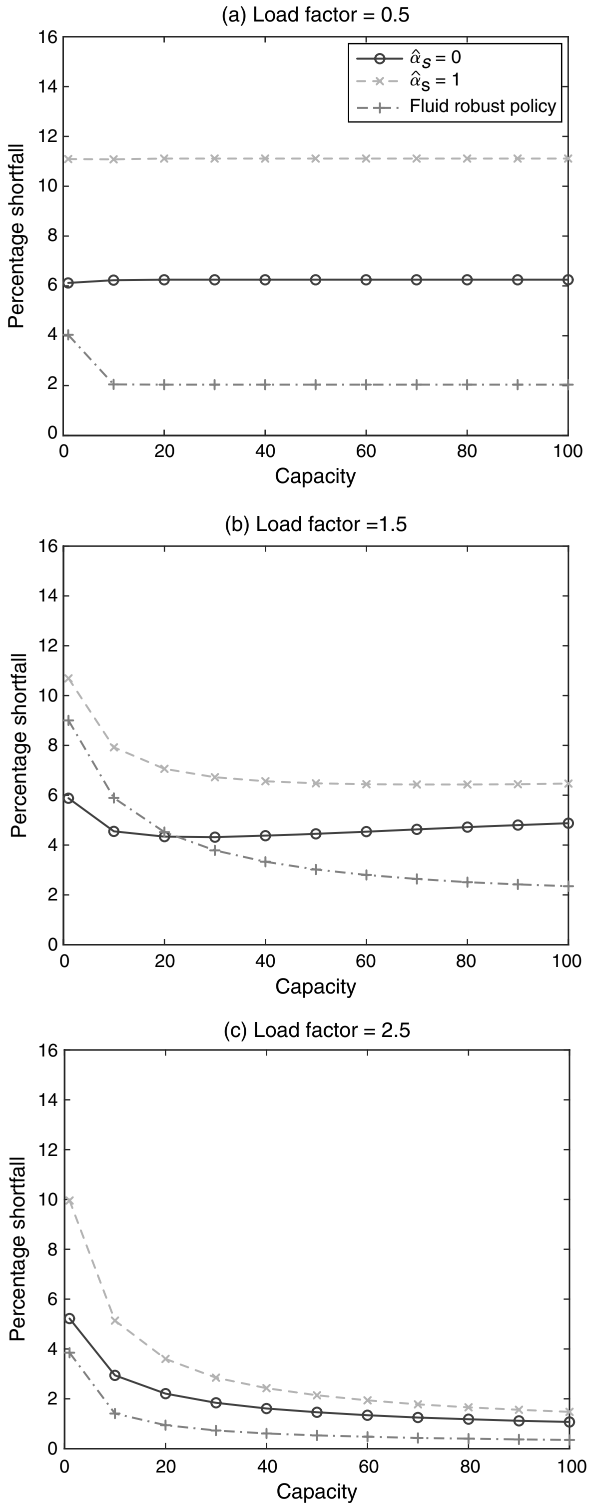
We consider regimes in which we scale up capacity c while maintaining constant load factors a/c. Figure 4 gives the maximum relative shortfall given the robust fluid prices for such regimes. For the sake of comparison, we also include the shortfalls of the optimal policies given naive assumptions that all customers are myopic ($\hat{\alpha}_s = 1$) and all customers are strategic ($\hat{\alpha}_s = 0$).

We observe that the robust policy always outperforms the naive but natural benchmark of assuming all customers are myopic. Assuming all customers are strategic yields smaller shortfall than assuming all customers are myopic, but the robust policy performs best for most instances. As we would expect given the asymptotic behavior of the problem, the stochastic problem behaves very similarly to the deterministic problem when capacity is loose (i.e., load factor is small), with the fluid robust policy achieving shortfalls around 2% and the benchmarks achieving shortfalls around 6% and 11%, respectively. In addition, the stochastic problem behaves similarly to the deterministic case as capacity and demand are scaled up proportionately. Therefore, we see that the robust fluid shortfalls converge to 2% or less for each of the plots in Figure 4. These are exactly the regions in the parameter space where our earlier analysis suggests that the deterministic model effectively proxies for the stochastic model. When capacity is tight (i.e., load factor is large), revenues are relatively insensitive to $\hat{\alpha}_s$ as we saw in the deterministic model in §4, and all three policies converge to near-full-information revenues when the load factor is set to 2.5.

We see that the performance of the fluid robust policy is relatively weak for modest load factors and small problem scalings, which are instances where our asymptotic analysis leaves open the possibility of significant differences between stochastic and deterministic problems. Because our fluid robust policy is computed assuming no rationing, it does not account for the heightened sensitivity of revenues to α induced by rationing. Indeed, we would expect the sensitivity of revenues to α to be most significant for tight capacities and moderate load factors.

We end this section with the observation that investing in learning α and making optimal use of this information may have the greatest impact when demand is stochastic, load factors are moderate, and problem scale is small. For other problem instances, revenues are either insensitive to α (when capacity is tight) or our robust policy effectively dampens the impact of uncertainty around α (when capacity is loose or the problem scale is moderate to large).

Figure 4 Maximum Percentage Shortfall of Robust Fluid Solution Compared with Stochastic Optimal Policies Assuming $\hat{\alpha}_s = 0$ and $\hat{\alpha}_s = 1$



We note that robust policy performs well as long as the problem scale is at least on the order of a few dozen. This is a very reasonable assumption in traditional RM contexts such as airplane and concert tickets, online retailing, and electronics. However, our analysis may be less applicable in a context such as brick-and-mortar fashion apparel retailing, where demand is fragmented among locations, colors, and sizes, and retailers typically only stock a few of each stock-keeping-unit per location.

6. Conclusion

A managerial implication of our work is that, while recognizing that the existence of strategic customer behavior is important, a seller can in many cases price reasonably well even without specific knowledge of strategic behavior. Our minimax pricing policy requires no assumptions on the true value of α yet yields revenues that are within a few percent of the full information revenues in the deterministic model (and asymptotically within a few percent of full information revenues in the stochastic problem). This is much better than if the seller ignored strategic customer behavior. At a high level, our approach has been to characterize the sensitivity of an objective to a hard-to-estimate parameter and to use robust optimization to obviate the estimation of that parameter. We believe that this idea could bear fruit in other contexts.

A broader take-away from our work is that while studying RM with full information can lead to valuable insights, ultimately RM optimization should not be considered independently of estimation issues. We are encouraged by the recent work of Besbes et al. (2010), who propose estimation procedures optimized for a specific pricing optimization problem, and of Levin et al. (2009b), who actively learn an aggregate demand process that allows for multiple sources of underlying uncertainty. We also applaud recent streams of research into robust decision making and active learning in RM.

Acknowledgments

The authors thank an anonymous associate editor and two anonymous referees for constructive comments that led to significant improvements in the paper. The authors also thank William L. Cooper for helpful comments on an earlier version of the manuscript. The work of Dan Zhang was partially supported by research grants from Natural Sciences and Engineering Research Council of Canada (NSERC) and Fonds de recherche sur la société et la culture (FQRSC) of Quebec. Adam Mersereau thanks the Kenan-Flagler Business School, University of North Carolina at Chapel Hill, for financial support. Portions of his research were supported by the Glen Raven Mills, Inc., Faculty Development Fund at the Kenan-Flagler Business School.

Appendix A. Asymptotic Relationships Between Deterministic and Stochastic Problems

In our model, demand in each period is upper bounded by a Poisson random variable with mean a. Therefore, if we let $c \to \infty$, the random variable $((c - N[\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta)])^+)/(N[\lambda_2(p_1, p_2, \hat{\alpha}_c, \theta)])$ approaches infinity for any p_1 and p_2, and in the limit when $c = \infty$,

$$f(p_1, p_2, \hat{\alpha}_c, \theta) = 1,$$

$$\mathsf{E}[S_1(p_1, p_2, \alpha, \theta)] = \mathsf{E}[N[\lambda_1(p_1, p_2, \alpha, \theta)]] = \lambda_1(p_1, p_2, \alpha, \theta),$$

and

$$\mathsf{E}[S_2(p_1, p_2, \alpha, \theta)] = \mathsf{E}[N[\lambda_2(p_1, p_2, \alpha, \theta)]] = \lambda_2(p_1, p_2, \alpha, \theta).$$

It follows that in the uncapacitated case, both the stochastic and deterministic problems become equivalent to the following:

$$\max_{a/b \geq p_1 \geq p_2 \geq 0} \{p_1 \lambda_1(p_1, p_2, \alpha, 1) + p_2 \lambda_2(p_1, p_2, \alpha, 1)\}. \tag{A1}$$

Therefore, we expect the deterministic problem to be a good approximation for the stochastic problem when capacity is large. Because customers do not need to estimate the fill rate θ, the customer estimate $\hat{\alpha}_c$ becomes irrelevant in this case.

We now turn our attention to a different asymptotic regime. Consider a sequence of problems indexed by a positive integer n, where for the nth problem the capacity and mean demand are both scaled up linearly with n. The expected fill rate for the nth problem is given by

$$f^n(p_1, p_2, \hat{\alpha}_c, \theta) = \mathsf{E}\left[\min\left\{1, \frac{(nc - N[n\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta)])^+}{N[n\lambda_2(p_1, p_2, \hat{\alpha}_c, \theta)]}\right\} \,\middle|\, N[n\lambda_2(p_1, p_2, \hat{\alpha}_c, \theta)] \geq 1\right].$$

Lemma 1. *Suppose $\hat{\alpha} \in [0, 1]$. As $n \to \infty$,*
(i) $(N[n\lambda_1(p_1, p_2, \hat{\alpha}, \theta)])/n \to \lambda_1(p_1, p_2, \hat{\alpha}, \theta)$ *almost surely,*
(ii) $(N[n\lambda_2(p_1, p_2, \hat{\alpha}, \theta)])/n \to \lambda_2(p_1, p_2, \hat{\alpha}, \theta)$ *almost surely,*
(iii) $f^n(p_1, p_2, \hat{\alpha}, \theta) \to \min\{1, (c - \lambda_1(p_1, p_2, \hat{\alpha}, \theta))^+/\lambda_2(p_1, p_2, \hat{\alpha}, \theta)\}$ *almost surely.*

For $t = 1, 2$, let $S_t^n(p_1, p_2, \alpha, \theta)$ and $S_t^n(p_1, p_2, \hat{\alpha}_c, \theta)$ indicate the estimated sales in period t for the seller and strategic customers, respectively. It immediately follows from Lemma 1 that

$$\frac{S_1^n(p_1, p_2, \alpha, \theta)}{n} \to \min\{c, \lambda_1(p_1, p_2, \alpha, \theta)\},$$

$$\frac{S_2^n(p_1, p_2, \alpha, \theta)}{n} \to \min\{(c - \lambda_1(p_1, p_2, \alpha, \theta))^+, \lambda_2(p_1, p_2, \alpha, \theta)\},$$

$$\frac{S_1^n(p_1, p_2, \hat{\alpha}_c, \theta)}{n} \to \min\{c, \lambda_1(p_1, p_2, \hat{\alpha}_c, \theta)\},$$

$$\frac{S_2^n(p_1, p_2, \hat{\alpha}_c, \theta)}{n} \to \min\{(c - \lambda_1(p_1, p_2, \hat{\alpha}_c, \theta))^+, \lambda_2(p_1, p_2, \hat{\alpha}_c, \theta)\}.$$

The following proposition follows directly from this observation.

Proposition 5. *Suppose capacity and demand in* (17) *are scaled up linearly. Then the stochastic problem* (17) *converges to the deterministic problem* (4).

The proof of part (i) is similar to that of Theorem 2 in Osadchiy and Vulcano (2010). Let $Y(t)$ denote a unit-rate Poisson counting process, then

$$\frac{N[n\lambda_1(p_1, p_2, \hat{\alpha}, \theta)]}{n} = \frac{Y(n\lambda_1(p_1, p_2, \hat{\alpha}, \theta))}{n} \to \lambda_1(p_1, p_2, \hat{\alpha}, \theta) \text{ almost surely},$$

where the convergence follows from the Strong Law of Large Numbers. Part (ii) of the lemma can be shown similarly.

To show part (iii), let $h^n(p_1, p_2, \hat{\alpha}, \theta) = \min\{1, (nc - N[n\lambda_1(p_1, p_2, \hat{\alpha}, \theta)])^+/(N[n\lambda_2(p_1, p_2, \hat{\alpha}, \theta)])\}$. We can then write

$$\begin{aligned}
f^n&(p_1, p_2, \hat{\alpha}, \theta) \\
&= \mathsf{E}\big[h^n(p_1, p_2, \hat{\alpha}, \theta) \,\big|\, N[n\lambda_2(p_1, p_2, \hat{\alpha}, \theta)] \geq 1\big] \\
&= \Big(\mathsf{E}\big[h^n(p_1, p_2, \hat{\alpha}, \theta)\big] \\
&\qquad - \mathsf{E}\big[h^n(p_1, p_2, \hat{\alpha}, \theta) \,\big|\, N\big[n\lambda_2(p_1, p_2, \hat{\alpha}, \theta)\big] = 0\big] \\
&\qquad\qquad \cdot \Pr\big(N\big[n\lambda_2(p_1, p_2, \hat{\alpha}, \theta)\big] = 0\big)\Big) \\
&\quad \cdot \big(\Pr\big(N[n\lambda_2(p_1, p_2, \hat{\alpha}, \theta)] \geq 1\big)\big)^{-1} \\
&= \frac{\mathsf{E}[h^n(p_1, p_2, \hat{\alpha}, \theta)] - \Pr(N[n\lambda_2(p_1, p_2, \hat{\alpha}, \theta)] = 0)}{\Pr(N[n\lambda_2(p_1, p_2, \hat{\alpha}, \theta)] \geq 1)}.
\end{aligned}$$

The desired result follows because

$$\lim_{n\to\infty} \Pr\big(N[n\lambda_2(p_1, p_2, \hat{\alpha}, \theta)] = 0\big) = 0,$$

$$\lim_{n\to\infty} \Pr\big(N[n\lambda_2(p_1, p_2, \hat{\alpha}, \theta)] \geq 1\big) = 1, \quad \text{and}$$

$$\begin{aligned}
\lim_{n\to\infty} &\mathsf{E}[h^n(p_1, p_2, \hat{\alpha}, \theta)] \\
&= \lim_{n\to\infty} \min\left\{1, \frac{(c - N[n\lambda_1(p_1, p_2, \hat{\alpha}, \theta)]/n)^+}{N[n\lambda_2(p_1, p_2, \hat{\alpha}, \theta)]/n}\right\} \\
&= \min\left\{1, \frac{(c - \lambda_1(p_1, p_2, \hat{\alpha}, \theta))^+}{\lambda_2(p_1, p_2, \hat{\alpha}, \theta)}\right\},
\end{aligned}$$

where we rely on the continuity of the min and $(\cdot)^+$ functions. □

We can therefore view problem (4) as a fluid approximation of problem (17). We have numerically explored this convergence. Each curve in the plots of Figure A.1 represents optimal expected revenue in the stochastic problem (expressed as a percentage of optimal revenue in the corresponding deterministic problem) as a function of capacity, where the demand parameters a are chosen to maintain specific load factors a/c. We set $b = a$ for each instance. The patterns are qualitatively consistent across different values of $\hat{\alpha}_c$. (In the deterministic version of the problem, we analytically show that $\hat{\alpha}_c$ is irrelevant when $\hat{\alpha}_c \leq \alpha$.) We see that expected revenue in the stochastic problem generally falls below that in the deterministic problem, reflecting a "cost of stochasticity." As predicted by Proposition 5, however, the difference between the deterministic and stochastic revenues becomes smaller as capacity and demand are scaled up. (Also, as predicted by Proposition 2, the revenue differences are the smallest when capacity is loose, i.e., load factor is small.) From the figure, we see that this convergence is quite fast. The differences are all less than 10% as long as $c \geq 20$, and the differences are less than 5% for $c = 100$. We believe that stocking quantities will be larger than 20 for many interesting practical instances. Plots of fill rates under optimal policies in the stochastic problem, presented in Figure A.2, reinforce that there

Figure A.1 Convergence to Fluid Revenue Under Various Customer Beliefs When True $\alpha = 0.2$

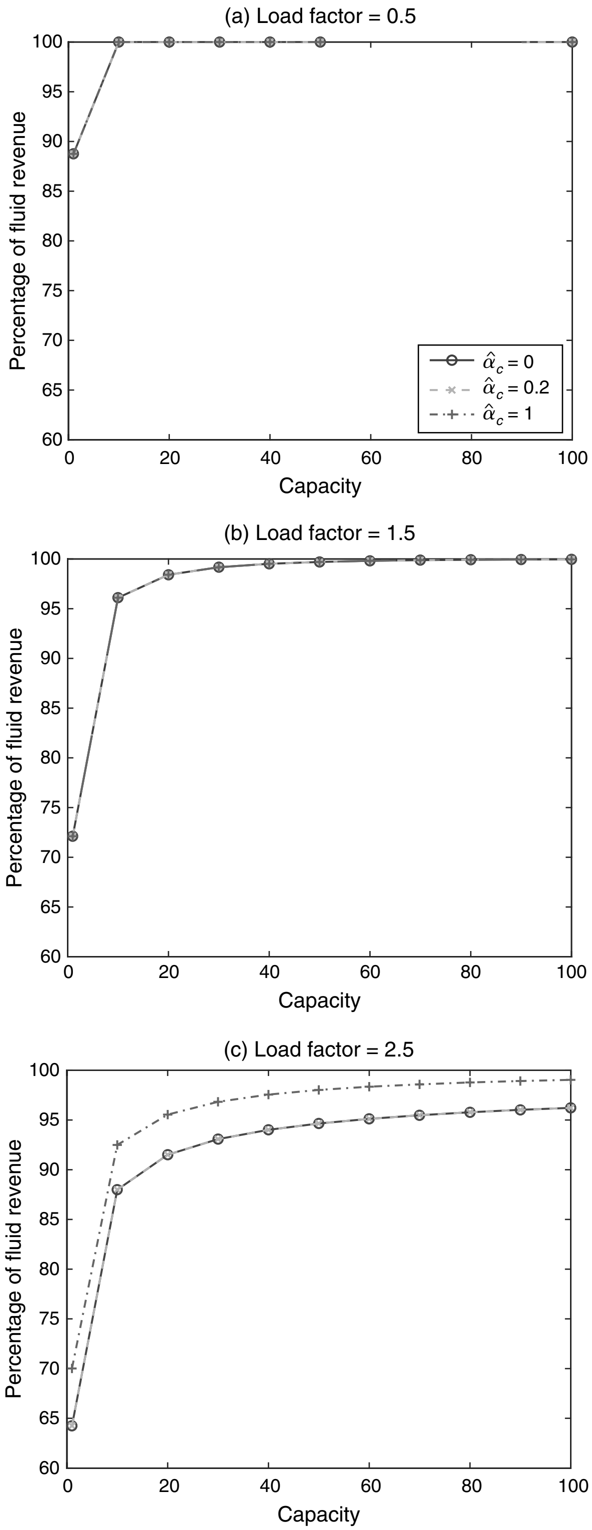
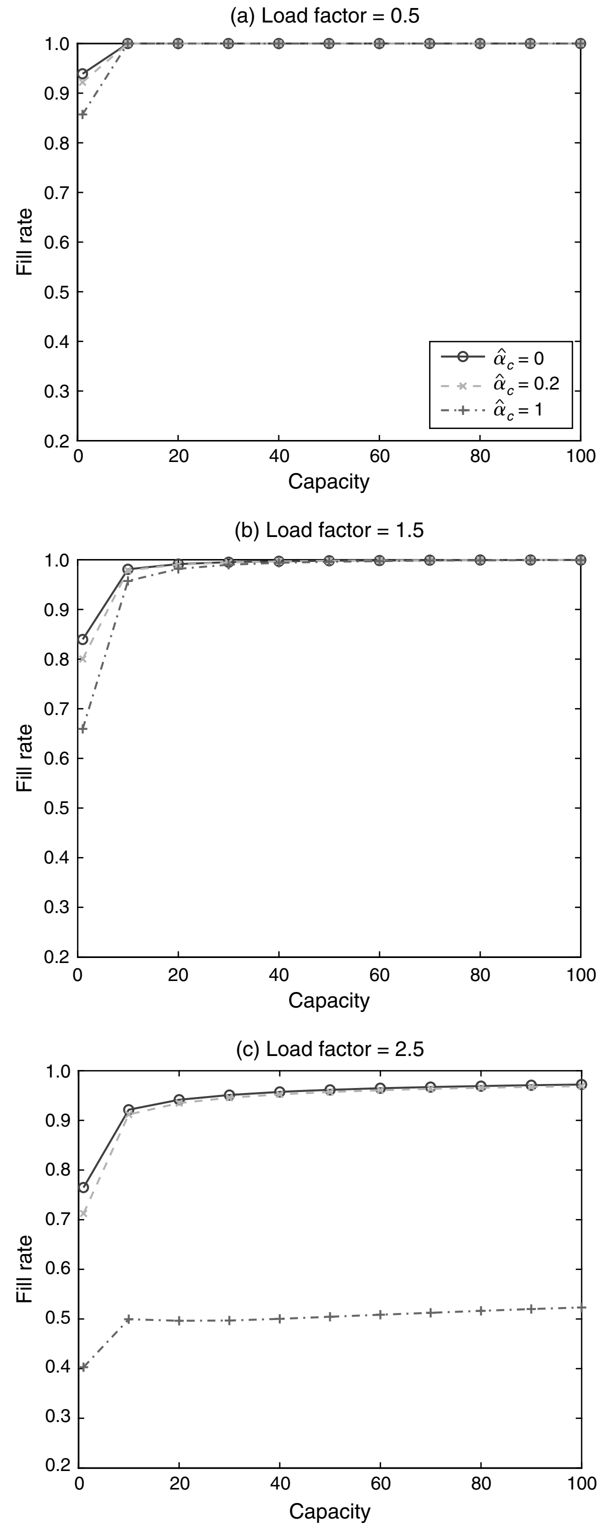


Figure A.2 Fill Rate Convergence Under Various Customer Beliefs When True $\alpha = 0.2$



is not a fundamental disconnect between the deterministic and stochastic problems. Fill rates in these plots converge smoothly and quickly to the deterministic optimal fill rates (which can be inferred from Proposition 1).

Appendix B. Proofs

Proof of Proposition 1
To solve problem (4), we consider two subproblems with additional constraints. In the first subproblem, we add the

constraint $a - bp_2 \le c$; in the second, we add the constraint $a - bp_2 \ge c$. The optimal objective value is the maximum of the optimal objective values of the two subproblems.

Subproblem with Additional Constraint $a - bp_2 \le c$**.** With the restriction $a - bp_2 \le c$, capacity is nonconstraining. Problem (4) with additional constraint $a - bp_2 \le c$ can be reduced to

$$\max_{p_1 \ge p_2 \ge 0,\, a - bp_2 \le c} \alpha(p_1 - p_2)(a - bp_1) + p_2(a - bp_2).$$

Note that θ is not involved above; because capacity is nonconstraining, we must have the optimal fill rate $\theta^I(\alpha, \hat{\alpha}_c, c) = 1$ at optimality. It can be easily shown that the optimal solution is given by

$$(p_1^I(\alpha, \hat{\alpha}_c, c), p_2^I(\alpha, \hat{\alpha}_c, c), \theta^I(\alpha, \hat{\alpha}_c, c)) = \begin{cases} \left(\dfrac{(3-\alpha)a}{(4-\alpha)b}, \dfrac{(2-\alpha)a}{(4-\alpha)b}, 1\right) & \text{if } c \ge \dfrac{2a}{4-\alpha}, \\ \left(\dfrac{2a-c}{2b}, \dfrac{a-c}{b}, 1\right) & \text{otherwise,} \end{cases} \tag{B1}$$

with corresponding optimal revenue

$$v^I(\alpha, \hat{\alpha}_c, c) = \begin{cases} \dfrac{a^2}{b(4-\alpha)} & \text{if } c \ge \dfrac{2a}{4-\alpha}, \\ \dfrac{(4a - (4-\alpha)c)c}{4b} & \text{otherwise.} \end{cases} \tag{B2}$$

Subproblem with Additional Constraint $a - bp_2 \ge c$**.** With the additional constraint $a - bp_2 \ge c$, we must have $\lambda_1(p_1, p_2, \alpha, \theta) \le c$ at optimality. This is because if $\lambda_1(p_1, p_2, \alpha, \theta) > c$, the objective value can be increased by raising p_1. Therefore, problem (4) can be rewritten as

$$\max_{a/b \ge p_1 \ge p_2 \ge 0,\, \theta} \{(p_1 - p_2)\lambda_1(p_1, p_2, \alpha, \theta) + p_2 c\} \tag{B3}$$

$$\text{s.t. } a - bp_2 \ge c, \tag{B4}$$

$$\lambda_1(p_1, p_2, \alpha, \theta) \le c, \tag{B5}$$

$$\theta = \frac{[c - \lambda_1(p_1, p_2, \hat{\alpha}_c, \theta)]^+}{a - bp_2 - \lambda_1(p_1, p_2, \hat{\alpha}_c, \theta)}. \tag{B6}$$

Note that we can, without loss of optimality, add the constraint $\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta) \le c$. To see this, note that if $\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta) \le c$, we must have $\theta = 0$ by (B6). It follows that (B5) reduces to $a - bp_1 \le c$. However, $\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta) > c$ implies $a - bp_1 > c$, a contradiction.

Next, we solve the problem (B3)–(B6) with the added constraint $\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta) \le c$. When $\hat{\alpha}_c \le \alpha$, the constraint $\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta) \le c$ is implied by $\lambda_1(p_1, p_2, \alpha, \theta) \le c$. Since $(c - \lambda_1(p_1, p_2, \hat{\alpha}_c, \theta))/(a - bp_2 - \lambda_1(p_1, p_2, \hat{\alpha}_c, \theta)) \ge (c - \lambda_1(p_1, p_2, \alpha, \theta))/(a - bp_2 - \lambda_1(p_1, p_2, \alpha, \theta))$, we obtain a relaxation of the problem by replacing (B6) with

$$\theta \ge \frac{c - \lambda_1(p_1, p_2, \alpha, \theta)}{a - bp_2 - \lambda_1(p_1, p_2, \alpha, \theta)},$$

which immediately implies

$$c - \lambda_1(p_1, p_2, \alpha, \theta) \le \theta\big[a - bp_2 - \lambda_1(p_1, p_2, \alpha, \theta)\big]. \tag{B7}$$

Using (B7) in the objective function (B3) and replacing the constraint on θ with $\theta \in [0, 1]$, we obtain a further relaxation as follows:

$$\max_{a/b \ge p_1 \ge p_2 \ge 0,\, \theta \in [0,1]} \{p_1 \lambda_1(p_1, p_2, \alpha, \theta) + p_2 \theta[a - bp_2 - \lambda_1(p_1, p_2, \alpha, \theta)]\} \tag{B8}$$

$$\text{s.t. } a - bp_2 \ge c, \tag{B9}$$

$$\lambda_1(p_1, p_2, \alpha, \theta) \le c. \tag{B10}$$

The objective function of the program above is bounded by (B2); for details, see Zhang and Cooper (2008). Because the solution (B1) satisfies constraints (B9) and (B10), the optimal solution when $\hat{\alpha}_c \le \alpha$ is the same as (B1) with corresponding objective value (B2).

Now, we turn to the case $\hat{\alpha}_c > \alpha$. In this case, we have $\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta) \ge \lambda_1(p_1, p_2, \alpha, \theta)$, therefore the constraint $\lambda_1(p_1, p_2, \alpha, \theta) \le c$ is implied by $\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta) \le c$. We need to solve the following problem:

$$\max_{a/b \ge p_1 \ge p_2 \ge 0,\, \theta} \{(p_1 - p_2)\lambda_1(p_1, p_2, \alpha, \theta) + p_2 c\} \tag{B11}$$

$$\text{s.t. } a - bp_2 \ge c, \tag{B12}$$

$$\lambda_1(p_1, p_2, \hat{\alpha}_c, \theta) \le c, \tag{B13}$$

$$\theta = \frac{c - \lambda_1(p_1, p_2, \hat{\alpha}_c, \theta)}{a - bp_2 - \lambda_1(p_1, p_2, \hat{\alpha}_c, \theta)}. \tag{B14}$$

Note that the constraint (B13) is implied by (B12) and (B14). Introduce the change of variable $r = (p_1 - p_2)/(1 - \theta)$. The program above can be rewritten as

$$\max_{a/b \ge p_1 \ge p_2 \ge 0,\, \theta \in [0,1]} \{r(1-\theta)\big[\alpha(a - bp_2 - br(1-\theta)) + (1-\alpha)(a - bp_2 - br)\big] + p_2 c\} \tag{B15}$$

$$\text{s.t. } a - bp_2 \ge c, \tag{B16}$$

$$\theta(a - bp_2) + (1-\theta)\big[\hat{\alpha}_c(a - bp_2 - br(1-\theta)) + (1-\hat{\alpha}_c)(a - bp_2 - br)\big] = c. \tag{B17}$$

The constraint (B17) can be simplified to

$$a - bp_2 - br(1-\theta)(1 - \hat{\alpha}_c\theta) = c. \tag{B18}$$

Using (B18) in the objective function (B15) and simplifying, we obtain the following optimization problem:

$$\max_{r \ge 0,\, \theta \in [0,1]} \left\{ r(1-\theta)\big[c - br\theta(1 - \alpha + (1-\theta)\hat{\alpha}_c)\big] - rc(1-\theta)(1 - \hat{\alpha}_c\theta) + \frac{c(a-c)}{b} \right\}. \tag{B19}$$

For a fixed θ, the objective function above is concave in r. From the first-order condition, at optimality r can be written as a function as θ as

$$r^*(\theta) = \frac{c\hat{\alpha}_c}{2b(1 - \alpha + (1-\theta)\hat{\alpha}_c)}. \tag{B20}$$

Plugging (B20) into (B19) leads to

$$\max_{\theta \in [0,1]} \{r^*(\theta)\theta(1-\theta)\big[c\hat{\alpha}_c - br^*(\theta)(1 - \alpha + (1-\theta)\hat{\alpha}_c)\big] + c(a-c)/b\}. \tag{B21}$$

An optimal θ can be determined by the first-order condition, which leads to the expression for $\theta^\dagger$ in the statement of the proposition. The expressions for optimal p_1 and p_2 can be found by noting that $p_1 = p_2 + r(1-\theta)$ and $p_2 = (a - c - br(1-\theta)(1-\hat{\alpha}_c\theta))/b$.

The overall optimal solution is determined by comparing the solutions of the two subproblems. □

Proof of Proposition 3

The relative shortfall can be written as

$$g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c) = \begin{cases} \left(\dfrac{\alpha-\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2 & \text{if } c \ge \dfrac{2a}{4-\hat{\alpha}_s} \ge \dfrac{2a}{4-\alpha}, \\ 1 - \dfrac{c(4-\alpha)[4a-(4-\alpha)c]}{4a^2} & \text{if } \dfrac{2a}{4-\hat{\alpha}_s} \ge c \ge \dfrac{2a}{4-\alpha}, \\ 0 & \text{if } \dfrac{2a}{4-\hat{\alpha}_s} \ge \dfrac{2a}{4-\alpha} \ge c, \\ \left(\dfrac{\alpha-\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2 & \text{if } c \ge \dfrac{2a}{4-\alpha} \ge \dfrac{2a}{4-\hat{\alpha}_s}, \\ 1 - \dfrac{4a^2(4-2\hat{\alpha}_s+\alpha)}{c[4a-(4-\alpha)c](4-\hat{\alpha}_s)^2} & \text{if } \dfrac{2a}{4-\alpha} \ge c \ge \dfrac{2a}{4-\hat{\alpha}_s}, \\ 0 & \text{if } \dfrac{2a}{4-\alpha} \ge \dfrac{2a}{4-\hat{\alpha}_s} \ge c. \end{cases}$$

The above can be simplified to

$$g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c) = \begin{cases} \left(\dfrac{\alpha-\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2 & \text{if } \alpha \le \hat{\alpha}_s \le 4 - \dfrac{2a}{c}, \\ 1 - \dfrac{c(4-\alpha)[4a-(4-\alpha)c]}{4a^2} & \text{if } \alpha \le 4 - \dfrac{2a}{c} \le \hat{\alpha}_s, \\ 0 & \text{if } 4 - \dfrac{2a}{c} \le \alpha \le \hat{\alpha}_s, \\ \left(\dfrac{\alpha-\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2 & \text{if } \hat{\alpha}_s \le \alpha \le 4 - \dfrac{2a}{c}, \\ 1 - \dfrac{4a^2(4-2\hat{\alpha}_s+\alpha)}{c[4a-(4-\alpha)c](4-\hat{\alpha}_s)^2} & \text{if } \hat{\alpha}_s \le 4 - \dfrac{2a}{c} \le \alpha, \\ 0 & \text{if } 4 - \dfrac{2a}{c} \le \hat{\alpha}_s \le \alpha. \end{cases} \tag{B22}$$

The minimax problem can be solved as follows:

$$\begin{aligned} &\min_{\hat{\alpha}_s\in[0,1]} \max_{\alpha\in[0,1]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c) \\ &= \min_{\hat{\alpha}_s\in[0,1]} \max\left\{ \max_{\alpha\in[0,\hat{\alpha}_s]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c),\ \max_{\alpha\in[\hat{\alpha}_s,1]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c) \right\} \\ &= \min\left\{ \min_{\hat{\alpha}_s\in[0,4-2a/c]} \max\left\{ \max_{\alpha\in[0,\hat{\alpha}_s]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c),\ \max_{\alpha\in[\hat{\alpha}_s,1]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c) \right\}, \right. \\ &\qquad \left. \min_{\hat{\alpha}_s\in[4-2a/c,1]} \max\left\{ \max_{\alpha\in[0,\hat{\alpha}_s]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c),\ \max_{\alpha\in[\hat{\alpha}_s,1]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c) \right\} \right\}. \end{aligned} \tag{B23}$$

To solve (B23), we consider three cases.

Case 1: $c < a/2$. It follows that $4 - 2a/c < 0$. Then (B23) reduces to

$$\min_{\hat{\alpha}_s\in[0,1]} \max\left\{ \max_{\alpha\in[0,\hat{\alpha}_s]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c),\ \max_{\alpha\in[\hat{\alpha}_s,1]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c) \right\}$$

From (B22), $g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c)$ takes the value 0 when $4 - 2a/c \le \alpha \le \hat{\alpha}_s$ or $4 - 2a/c \le \hat{\alpha}_s \le \alpha$. It follows that the objective value is 0.

Case 2: $c > 2a/3$. It follows that $4 - 2a/c > 1$. Then (B23) reduces to

$$\begin{aligned} &\min_{\hat{\alpha}_s\in[0,1]} \max\left\{ \max_{\alpha\in[0,\hat{\alpha}_s]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c),\ \max_{\alpha\in[\hat{\alpha}_s,1]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c) \right\} \\ &\quad = \min_{\hat{\alpha}_s\in[0,1]} \max\left\{ \max_{\alpha\in[0,\hat{\alpha}_s]} \left(\frac{\alpha-\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2,\ \max_{\alpha\in[\hat{\alpha}_s,1]} \left(\frac{\alpha-\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2 \right\} \\ &\quad = \min_{\hat{\alpha}_s\in[0,1]} \max_{\alpha\in[0,1]} \left(\frac{\alpha-\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2 = \frac{1}{49} \approx 2\%. \end{aligned}$$

The minimizer in the last step above is $\hat{\alpha}_s = 1/2$.

Case 3: $a/2 \le c \le 2a/3$. In this case $0 \le 4 - 2a/c \le 1$. Using (B22) and noting that in some cases $g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c)$ takes the value 0, we obtain

$$\begin{aligned} &\min_{\hat{\alpha}_s\in[0,1]} \max_{\alpha\in[0,1]} g^{\mathrm{R}}(\alpha, \hat{\alpha}_s, c) \\ &= \min\left\{ \min_{\hat{\alpha}_s\in[0,4-2a/c]} \max\left\{ \max_{\alpha\in[0,\hat{\alpha}_s]} \left(\frac{\alpha-\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2,\ \max_{\alpha\in[\hat{\alpha}_s,4-2a/c]} \left(\frac{\alpha-\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2, \right. \right. \\ &\qquad \left. \max_{\alpha\in[4-2a/c,1]} 1 - \frac{4a^2(4-2\hat{\alpha}_s+\alpha)}{c[4a-(4-\alpha)c](4-\hat{\alpha}_s)^2} \right\}, \\ &\qquad \left. \min_{\hat{\alpha}_s\in[4-2a/c,1]} \max_{\alpha\in[0,4-2a/c]} 1 - \frac{c(4-\alpha)[4a-(4-\alpha)c]}{4a^2} \right\} \\ &= \min\left\{ \min_{\hat{\alpha}_s\in[0,4-2a/c]} \max\left\{ \max_{\alpha\in[0,4-2a/c]} \left(\frac{\alpha-\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2, \right. \right. \\ &\qquad \left. \max_{\alpha\in[4-2a/c,1]} 1 - \frac{4a^2(4-2\hat{\alpha}_s+\alpha)}{c[4a-(4-\alpha)c](4-\hat{\alpha}_s)^2} \right\}, \\ &\qquad \left. \min_{\hat{\alpha}_s\in[4-2a/c,1]} \max_{\alpha\in[0,4-2a/c]} 1 - \frac{c(4-\alpha)[4a-(4-\alpha)c]}{4a^2} \right\}. \end{aligned} \tag{B24}$$

Since $[(\alpha-\hat{\alpha}_s)/(4-\hat{\alpha}_s)]^2$ is convex in α,

$$\max_{\alpha\in[0,4-2a/c]} \left(\frac{\alpha-\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2 = \max\left\{ \left(\frac{\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2, \left(\frac{4-2a/c-\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2 \right\}.$$

It can be verified that $1 - (4a^2(4-2\hat{\alpha}_s+\alpha))/(c[4a-(4-\alpha)c](4-\hat{\alpha}_s)^2)$ is increasing in α for $\hat{\alpha}_s \in [0, 4-2a/c]$ and $a/2 \le c \le 2a/3$. Therefore,

$$\max_{\alpha\in[4-2a/c,1]} 1 - \frac{4a^2(4-2\hat{\alpha}_s+\alpha)}{c[4a-(4-\alpha)c](4-\hat{\alpha}_s)^2} = 1 - \frac{4a^2(5-2\hat{\alpha}_s)}{c(4a-3c)(4-\hat{\alpha}_s)^2}.$$

Notice that the last term within bracket in (B24) does not involve $\hat{\alpha}_s$, and it is easy to show that it is maximized at

$\alpha = 0$ resulting in a value of $1 - 4c(a-c)/a^2$. Combining the above, (B24) simplifies to

$$\min\left\{\min_{\hat{\alpha}_s \in [0,\,4-2a/c]} \max\left\{\left(\frac{\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2, \left(\frac{4-2a/c-\hat{\alpha}_s}{4-\hat{\alpha}_s}\right)^2, 1 - \frac{4a^2(5-2\hat{\alpha}_s)}{c(4a-3c)(4-\hat{\alpha}_s)^2}\right\}, 1 - \frac{4c(a-c)}{a^2}\right\}$$
$$= \left(\frac{4(3c-a)(a-c)-a^2}{4(3c-a)(a-c)+a^2}\right)^2.$$

In the above, the minimization occurs at $\hat{\alpha}_s = 2 - [a^2/(2\cdot(3c-a)(a-c))]$. This completes the proof. □

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