A Model of Customer Reward Programs with Finite Expiration Terms*

Yacheng Sun
School of Economics and Management
Tsinghua University
sunyc@sem.tsinghua.edu.cn
Dan Zhang
Leeds School of Business
University of Colorado Boulder
dan.zhang@colorado.edu

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Abstract

A little understood phenomenon of customer reward programs is the prevalent use of finite reward expiration terms. We develop a theoretical framework to investigate the economic rationale behind this phenomenon, and the tradeoff between short and long expiration terms. In our model, a monopolistic firm sets the expiration term, along with the price and reward size, and interacts with consumers over an infinite horizon. Consumers are heterogeneous in shopping probabilities and product valuations, and forward-looking in making purchase decisions. We find that a customer reward program with a finite expiration term can increase firm profits when (1) the valuation difference within the consumer population is *intermediate* and (2) the shopping probabilities and valuations are *negatively* correlated among consumers. Several model extensions confirm the robustness of these results. Finally, we conduct an empirical investigation on the reward program practice of the top 100 US retailers, which provides directional support for several key theoretical predictions.

Keywords: customer reward program, expiration term, reward structure, forward-looking consumers, dynamic programming.

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1 Introduction

Customer reward programs (hereafter "reward programs") are commonly used by US retailers. Our own survey finds that 55 of the top 100 US retailers offer some form of reward programs.[1] An interesting phenomenon of reward programs is the prevalent use of finite reward expiration terms, a date after which earned rewards are lost if not redeemed. Among these 55 reward programs, 47 stipulate a finite expiration term, which can be as short as a month (e.g., Safeway) or as long as multiple years (e.g., Gap).

Apparently a deliberate decision made by firms, finite expiration terms have a nontrivial impact on consumers and firms alike. When consumers evaluate a reward program, they account not only for the face value of the rewards, but also the likelihood of utilizing such rewards (O'Brien and Jones, 1995). A finite reward expiration term acts as a redemption hurdle, and therefore changes the intertemporal tradeoff that the reward program induces for consumers.[2] Intuitively, a shorter (longer) expiration term reduces (raises) the likelihood of redeeming the reward, and hence decreases (increases) the value of the reward. A finite expiration term also entails a nontrivial tradeoff for the firm, for which reward points are considered as outstanding debts (Chun et al., 2015). While a shorter expiration term can increase *breakage* (the percentage of unredeemed rewards) and lower the firm's costs of honoring the rewards, it also reduces consumers' perceived value of the rewards, diminishing the firm's ability to charge higher prices and attract consumers.

There is an extensive literature on reward programs investigating the rationales for offering customer rewards (e.g., Bakhtiari et al., 2013; Deighton, 2000; Lewis, 2004; Liu, 2007; Rust et al., 2001; Winer, 2001) and how to improve the design of reward programs (e.g., Chun and Ovchinnikov, 2015; Kopalle et al., 2012; Stourm et al., 2015). Little research has formally considered finite reward expiration terms. Kopalle and Neslin (2003) assume that rewards expire after exactly one period, and therefore are not amenable to analyzing the length of expiration terms. A notable exception is Hartmann and Viard (2008), who show that finite expiration terms can create switching costs for consumers; however, they study existing members of a reward program, and they do not examine how an optimal expiration term should be determined.

Consequently, several interesting research questions remain: When should a firm set a short

[1] See the Online Appendix for details.

[2] There are other types of redemption hurdles. For example, reward availability is important in some industries (e.g., reward seat availability for airline frequent flyer miles programs; see recent papers by Lu and Su (2015) and Chun and Ovchinnikov (2015)). This paper focuses on the retail industry, for which reward availability is less of an issue. For the retailers we surveyed, the most common reward offered is store credit, which is readily available for redemption.

vs. a long reward expiration term? How should the firm set price and reward size, along with the expiration term? What are the effects of reward expiration terms on firm profitability, consumer welfare, and social welfare?

This paper answers these questions in a succinct theoretical framework. Specifically, we model a monopolistic firm that interacts with its consumers over an infinite horizon. The firm is forward-looking, and seeks to maximize long-run average profit by making three decisions that jointly define a reward program: price, reward size, and reward expiration term. Consumers are strategic in making purchase decisions in order to maximize long-run average surpluses. Our model accounts for two types of consumer heterogeneity — *shopping probabilities* and *valuations* — with a parsimonious, yet flexible parametrization that allows an arbitrary correlation between these two characteristics.

Our analysis leads to several interesting findings. First, a finite expiration term gives the firm a lever to price discriminate based on consumers' shopping probabilities. Infrequent consumers redeem less, and consequently pay higher effective prices. Furthermore, a finite expiration term can also help the firm retain consumers with low valuations and high shopping probabilities. This insight helps us better understand reward breakage, for which accounting and marketing departments may have opposite views; see also a relevant discussion in Stourm et al. (2015). Our result implies that a positive but not excessive level of breakage can be *desirable.*

Second, the optimal expiration term depends on the consumer heterogeneity in both product valuations and shopping probabilities. In particular, the optimal expiration term increases in consumer valuation difference. Under an optimal reward program, frequent consumers always pay a lower effective price. The reward program is designed to "squeeze" high-valuation-infrequent consumers. Intuitively, a longer expiration term is needed to separate out high- and low-valuation consumers for a large valuation difference. In contrast, when the shopping probability difference is high, a short expiration term is sufficient. Somewhat surprisingly, the optimal reward size does not necessarily monotonically decrease in the optimal length of the reward expiration term, as the optimal price tends to increase with the length of the reward expiration term, partially offsetting any increase in rewards.

Third, the effectiveness of a reward program depends crucially on market composition. First, consumers' shopping probability and valuation for the product must be *negatively* correlated. Since the effectiveness of a reward program stems from its ability to segment high-valuation-infrequent consumers from low-valuation-frequent consumers, it is profit-enhancing only when the former segment is relatively large. Second, the consumer valuation difference must be intermediate. An implication is that firms with a relatively homogeneous consumer population are less likely to adopt reward programs.

Finally, we study consumer and social welfare. While an optimal reward program never hurts frequent consumers, it has an ambiguous effect on infrequent consumers. On the one hand, a reward program enables the firm to charge infrequent consumers higher effective prices. On the other hand, a reward program may allow the firm to retain low-valuation-frequent consumers who will not be served otherwise. The impact of a reward program on social welfare is also ambiguous.

Our model produces several empirically testable observations. First, the length of the expiration term decreases with the heterogeneity in shopping probability. Second, the length of the optimal expiration term monotonically increases with the profit margin, but is non-monotonic with respect to the optimal reward size. We conduct an empirical investigation on the reward program practice of the top 100 US retailers identified by the National Retailer Federation. Our data combine reward program and firm characteristics collected and compiled from publicly available sources, as well as primary survey data on the purchase frequencies of 2,500 consumers. The results from the empirical analyses confirmed the validity of our main observations and in turn, provide useful guidelines for retailers to set the expiration terms for their reward programs.

We consider several model extensions to check the robustness of our theoretical results. First, we consider a model with *rush redemption* of rewards; that is, consumers increase their shopping probabilities as their rewards are close to expiration. We show that our results from the main analysis hold qualitatively. Interestingly, rush redemption does not necessarily decrease the seller's profit. Second, the ability to customize rewards based on shopping probability does not necessarily increase firm revenue. Third, we consider an extension with a nontrivial outside option for consumers, which can be viewed as an implicit modeling of competition (e.g., Kopalle et al., 2012; Lewis, 2004). Our analysis reveals that as the outside option becomes more attractive, i) consumers are better off, ii) for a fixed expiration term, a lower reward is offered together with a lower price, iii) the optimal expiration term is longer, and iv) the firm has a stronger incentive to use a reward program. Fourth, we discuss the implication of efficient rewards (Kim et al., 2001). Not surprisingly, more efficient rewards make reward programs more appealing to sellers. Finally, we show that with consumer discounting, a larger reward is needed to induce consumer purchases. Hence, the firm's profits will be lower. All of our results hold qualitatively otherwise.

Our research contributes to the growing operations literature in reward programs (Chun et al., 2015; Chun and Ovchinnikov, 2015; Lu and Su, 2015). Like Lu and Su (2015) and Chun and

Ovchinnikov (2015), we study a monopolistic firm and its interaction with forward-looking consumers. Similar to these studies, we endogenously derive consumers' demand as a function of the design components of a reward program. However, the tension in Lu and Su (2015) and Chun and Ovchinnikov (2015) stems from the limited availability of rewards, whereas in our model, there is unlimited reward availability and customers face the threat of expiring rewards. Note that our main focus is in the retail setting, as opposed to a capacitated revenue management setting. An examination of the reward programs operated by the top 100 US retailers shows that the most frequently used reward is store credit, for which there is practically no capacity limit.

Reward programs as a price-discrimination mechanism has been studied in the earlier work of Kim et al. (2001) and Hartmann and Viard (2008). Kim et al. (2001) consider a model with heavy users and light users and point out that reward programs can be used to price discriminate light users. In their model, only heavy users are eligible for rewards. Reward expiration is irrelevant in their two-period setup. Using the data from a golf course, Hartmann and Viard (2008) structurally estimate consumer switching costs, and they conjecture that reward programs price discriminate in favor of frequent customers. Our study provides a theoretical validation for their conjecture and identifies an important boundary condition for the effectiveness of reward programs, regarding the underlying (vs. observed) consumer composition.

While our work proposes a normative model in which breakage is the only driver for reward underutilization, several authors offer behavioral explanations for such an intriguing phenomenon. Shu and Gneezy (2010) show that people procrastinate in redeeming gift certificates and gift cards with long deadlines more than those with short deadlines. Stourm et al. (2015) construct and empirically test a structural model of redemption choice that unifies the economic, cognitive, and psychological motivations for reward points stockpiling. While we do not incorporate behavioral considerations in our model, we certainly view them as an important next step.

There is a large body of operations literature that considers strategic customer behavior (e.g., Su, 2007; Su and Zhang, 2008, 2009; Aviv and Pazgal, 2008; Yin et al., 2009; Aviv and Wei, 2014; Besbes and Lobel, 2015). Customers in these papers make at most one purchase and are forward-looking in that they strategically time their purchases. Different from much of this literature, customers in our model can make repeated purchases from the seller. Customers in our model trade off the surplus from the immediate purchase and the expected value of the earned rewards, but do not intentionally delay purchases or stockpile (Su, 2010). Since price is constant in our model, purchase timing will not improve the consumer surplus. The assumption of no stockpiling is reasonable for most services and perishable products, such as groceries or freshly brewed coffee.

The remainder of the paper is organized as follows. Section 2 describes the model. Sections 3 and 4 analyze the consumers' and the firm's decision problems, respectively. Section 5 presents the results of the empirical investigation. Section 6 considers several model extensions. Section 7 concludes with a summary of the results, limitations, and future research directions. The Online Appendix provides additional analysis and all proofs.

2 Model Setup

We consider a monopolistic firm that offers a single, non-durable product over infinitely many periods. Without loss of generality, the marginal cost is normalized to zero, so we use revenue and profit interchangeably. Consumers are infinitesimal and the total market size is normalized to 1.

2.1 Market Composition

The effects of an expiration term differ across consumers. Intuitively, frequent shoppers are less likely to be inconvenienced by a short expiration term, because they are likely to redeem the reward before it expires. Furthermore, consumers who have high valuation for the product are more likely to purchase, even if the expiration term is short. Thus, we incorporate shopping probability and product valuation as the two key dimensions of consumer heterogeneity.[3]

In each period, each consumer comes to the market with a fixed probability, referred to as *shopping probability.* A fraction $\beta \in (0, 1)$ of the consumers have shopping probability λ_F, and the remaining (1-β) consumers have the shopping probability λ_I, where $0 \leq \lambda_I < \lambda_F \leq 1$. Consumers with shopping probabilities λ_F and λ_I are referred to as frequent and infrequent consumers, respectively. Following Kim et al. (2001), we assume that a consumer's shopping probability is exogenous (e.g., a grocery shopper does not invent a shopping trip just to redeem the rewards).

A consumer in the market chooses whether or not to buy a single unit of the product from the focal firm. Each consumer's time-invariant valuation for the product is denoted by the parameter $v \in \{v_H, v_L\}$. A fraction α of the consumers has the valuation v_H and the rest have the valuation v_L, with $v_L < v_H$. To avoid triviality, we assume $\alpha < 1$. Consequently, consumers can be classified

[3]Our emphasis on valuation and shopping probability follows existing research in reward programs, such as Kim et al. (2001), who segment consumers into heavy vs. light users (equivalent to high vs. low shopping probability) and different transportation costs (similar to product valuations). This emphasis is also consistent with industry practice, where these two characteristics are often used to segment consumers. See, for example, http://www.nielsen.com/content/dam/corporate/us/en/newswire/uploads/2009/06/segmentation-and-customer-loyalty-white-paper.pdf. Retrieved on March 20, 2017.

into four homogeneous segments: high-valuation-frequent (HF), high-valuation-infrequent (HI), low-valuation-frequent (LF), and low-valuation-infrequent (LI). The four segments are illustrated in Figure 1.[4]

Figure 1: The four consumer segments

	λ_F (β)	λ_I (1-β)
v_H (α)	HF (γ)	HI (α-γ)
v_L (1-α)	LF (β-γ)	LI (1-α-β+γ)

We use γ to denote the size of the high-valuation-frequent consumers. Note that γ can take any value between 0 and $\min\{\alpha, \beta\}$. This parametrization is succinct, yet allows for any possible correlation between the consumer valuation and shopping probability. Formally, let ρ denote this correlation:

$$\rho = \frac{\gamma - \alpha\beta}{\sqrt{\alpha(1-\alpha)}\sqrt{\beta(1-\beta)}}.$$

When $\gamma = \alpha\beta$, the two consumer characteristics are independent. When $\gamma > (<) \alpha\beta$, the two consumer characteristics are positively (negatively) correlated.[5]

2.2 Structure of the Reward Program

The firm's objective is to maximize long-run average revenue. In the absence of other marketing instruments, the firm uses price as the only marketing lever. Since market demand is stationary over time, the firm should charge the same price in each period in order to maximize revenue. However, it is possible that the firm can further increase revenue by introducing a reward program

[4]It can be argued that our choice of two groups along each dimension of consumer characteristics is not comprehensive. Consumers are often segmented into more than two groups, based on certain characteristics in practice. For example, Best Buy classifies its best customers into five segments, where "Barry" is a male with a six-figure income who purchases whatever he wants, regardless of the cost. Our model, however, can be easily adapted to situations with multiple groups along each consumer characteristic. Indeed, the major component of our analysis, the consumers' decision problem analyzed in Section 3, is conducted for any pair of valid consumer valuation and shopping probability.

[5]The correlation between these two consumer characteristics has been investigated in industry surveys and academic studies. For example, an online survey by The Hartman Group in 2013 asked US households of different income groups regarding how often they went grocery shopping in a store. The survey shows that 19 percent of US households with an income over 75,000 US dollars went grocery shopping two to three times a week, while 19 percent of households with an income less than 40,000 US dollars purchased grocery items once a month or less. Income levels can be viewed as a proxy for consumer valuations in our model. The findings from Kim and Rossi (1994) suggest a negative correlation among grocery shoppers (frequent shoppers tend to be more price sensitive).

with reward size r and expiration term K. The reward r is interpreted as the perceived benefit that consumers derive, net the hassle costs for redeeming the reward. The actual reward can take various forms, including price discounts, service upgrades, free products, and so on. The reward cannot be used to offset the price paid in the current transaction; instead, it can be redeemed in the next K periods when the consumer purchases again. The distinguishing feature of the reward program in our model is the expiration term K.

Let p be the price charged by the firm, which is constant over time.[6] Naturally, we have $r \leq p$. The firm's strategy can be represented by the triplet (p, r, K). Note that a triplet $(p, 0, K)$ represents the case without a reward program, where K is irrelevant.

We make several assumptions for the reward programs considered in our model. First, the only way consumers can redeem rewards is through future purchases. Using the terminology of Zhang et al. (2000), the firm employs a *rear-loaded* incentive. This assumption is informed by the most common design of reward programs: out of the fifty-five reward programs examined earlier, thirty-nine (71%) programs issue rewards as store credit, which can only be used for future purchases. Second, the same reward program (p, r, K) is offered to all consumers. In Section 6.2, we examine the scenario where the firm customizes rewards based on shopping probability. Third, past purchase is used as the *only* criterion to determine whether a consumer qualifies for rewards.[7] This assumption implies that no other criteria, such as knowledge of the consumers' demographic characteristics and price sensitivities, are used.[8] This assumption is consistent with the practice of many reward programs.[9] Fourth, we assume that the reward is always available, which is a reasonable assumption for the retail industry, but may be less applicable for capacitated industries.[10] Finally, our model does not consider reward accumulation and redemption thresholds. In principle, relaxing any of the assumptions above augments the decision space of the firm, which can only

[6]Our model accommodates situations where retail prices are relatively stable over time, e.g., when a retailer offers "everyday low prices," which is a popular retail pricing strategy. Even in cases where prices are varied systematically over time, price changes are often driven by factors other than outstanding customer rewards.

[7]Programs that issue rewards based only on past purchases are referred to as *Type 3* reward programs in Berman (2006). These are popular for firms that include credit card companies (e.g., Citibank PremierPass Elite, Discover Card: The Miles Card, AmEx Membership Rewards, Diners Club Rewards), office supply firms (e.g., Office Depot Advantage and Staples Business Rewards programs) and consumer electronics stores (e.g., BestBuy Reward Zone Program).

[8]Some reward programs do use such information to create customized pricing and product offers. These are referred to as *Type 4* programs in Berman (2006), and examples include Harris Teeter's E-VIC program and Harrah's Total Reward program.

[9]For example, Staples states that in order to earn rewards, the consumer simply places an order online, or makes a purchase in a Staples retail store and provides the cashier with his/her Rewards number when checking out. For any purchase he/she make (excluding postage stamps, phone/gift cards, savings passes), he/she earns up to 5% back in rewards."

[10]For example, in the airline industry, reward seat availability is not guaranteed and has been a major cause of consumer complaints.

lead to higher firm profits. Therefore, our model establishes a useful benchmark and provides an interesting result that a reward program can be profitable, even when it is not conditioned on any other criteria except recency.

3 The Consumers' Problem

We first analyze the decision problem of a representative consumer with valuation $v \in \{v_H, v_L\}$ and shopping probability $\lambda \in \{\lambda_{\mathrm{F}}, \lambda_{\mathrm{I}}\}$. Consistent with substantial evidence from the existing empirical literature (Lewis, 2004; Kopalle et al., 2012), consumers are forward-looking and account for their future reward redemption when making purchase decisions. We assume no consumer discounting over time, and the consumer's objective is to maximize long-run average consumer surplus. In Online Appendix C, we analyze an extension with consumer discounting.

Given the firm's reward program (p, r, K), the consumer's problem can be formulated as an infinite-horizon average-reward stochastic dynamic program (Ross, 1983). Suppose the consumer comes to the market at the beginning of a period. If a reward is available (i.e., has not expired), she will use the reward immediately because there is no incentive for her to save the reward for a later occasion. If there is no reward, she either pays the full price or chooses not to purchase. If a purchase is made, she earns a *new* reward, which is good for the next K periods. It follows that at any given time, the consumer has either zero or one reward, and her pay-off-relevant state variable is the number of periods until the reward expiration, denoted by $i \in \{0, 1, ..., K\}$. The state $i = 0$ denotes the state with no available reward.

Following the conventions for infinite-horizon average-reward stochastic dynamic programs (Ross, 1983), the consumer's value function consists of two parts. The first part, g^*, is the optimal average reward, or the optimal average per period consumer surplus. The second part, $h(\cdot)$, is the bias function, or the deviation from the average reward. The optimality equations can be written as

$$g^* + h(i) = \begin{cases} \lambda \max\{v - p + r + h(K), h(i-1)\} + (1-\lambda)h(i-1), & \text{if } i = 1, \dots, K, \\ \lambda \max\{v - p + h(K), h(0)\} + (1-\lambda)h(0), & \text{if } i = 0, \end{cases} \quad \forall i = 0, 1, \dots, K. \tag{1}$$

Equation (1) states that if a consumer in state $i \geq 1$ comes to the market and makes a purchase, the consumer derives an immediate surplus $v - p + r$, and earns a new reward with expiration term

K (hence the new state K).[11] Otherwise she earns zero surplus, and the next state becomes $i-1$; that is, her reward is one period closer to expiration. The dynamics for a consumer with no reward (state $i = 0$) is similar, except that the consumer pays the full price p, if a purchase is made. One can easily show that $h(i) \geq h(i-1)$ for all $i \geq 1$; intuitively, a reward with a longer expiration never hurts the consumer.

Observe that a consumer will contribute to the firm's long-run average revenue only if she purchases in the state with no reward ($i = 0$). Otherwise, the consumer exits the market in the long run. From the optimality equation (1), a consumer in state 0 purchases if and only if the immediate surplus from purchasing without a reward, $v - p$, plus the continuation value of a new reward, $h(K)$, is higher than the continuation value of having no reward, $h(0)$; i.e.,

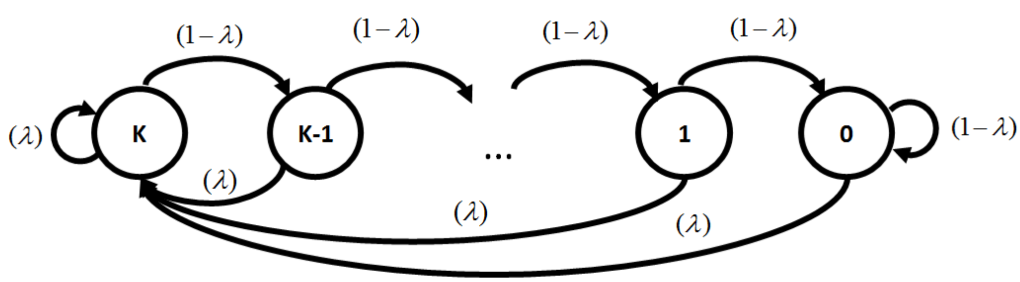
$$v - p + h(K) \geq h(0) \quad \Leftrightarrow \quad h(K) - h(0) \geq p - v,$$

Similarly, a consumer in state $i \geq 1$ purchases if and only if

$$v - p + r + h(K) \geq h(i-1) \quad \Leftrightarrow \quad h(K) - h(i-1) \geq p - v - r,$$

where $v-p+r$ is the immediate surplus of purchasing with a reward. Lemma 2 in Online Appendix A derives the consumer's bias function and establishes that if it is optimal for a consumer to purchase without a reward (in state 0), then it is also optimal for her to purchase with a reward (in states $1, ..., K$). The state of a consumer who purchases every time she comes to the market follows a Markov chain illustrated in Figure 2.

Figure 2: Reward state transitions



We have the following proposition, which we choose to fix $h(0) = 0$, since the solution for $h(\cdot)$

[11] Our analysis assumes that the same reward r is earned when a purchase is made, but regardless of out-of-pocket expense, which is p or $p - r$. This assumption is justifiable when r is small relative to p. In most reward programs, the reward from each purchase is less than 5% of the purchase price.

is not unique in (1).

Proposition 1. *For a consumer, if the valuation v is higher than the effective price, $p - r\left[1-(1-\lambda)^K\right]$, then a solution to the optimality equations (1) is given by*

$$h(i) = r\left[1-(1-\lambda)^i\right], \qquad \forall i = 0, 1, \ldots, K, \tag{2}$$

$$g^* = \lambda\left(v - p + r\left[1-(1-\lambda)^K\right]\right). \tag{3}$$

It is optimal for a consumer with the valuation v and shopping probability λ to purchase in each state. Let q_i denote the stationary probability of state i. We have

$$q_i = \begin{cases} (1-\lambda)^K, & \text{if } i = 0, \\ \lambda(1-\lambda)^{K-i}, & \text{if } i = 1, \ldots, K, \end{cases} \qquad \forall i = 0, \ldots, K. \tag{4}$$

We make a few observations regarding the optimal average reward g^* and the bias function $h(\cdot)$. The optimal average surplus g^* is increasing in λ, v, r, and K, and decreasing in p. Intuitively, the average surplus is higher if the consumer shops more frequently and/or has higher valuation. It is also higher if the reward amount is higher or the expiration term is longer. The bias function $h(i)$ only depends on r, λ, and i, and is increasing in all three. The fact that $h(i)$ is increasing in i implies that a reward with a longer expiration term is more valuable.[12]

The stationary probabilities characterized in equation (4) have straightforward interpretations. Note that with a purchase, a consumer earns a reward r redeemable in the next K periods. Since the consumer purchases whenever she comes to the market, the probability of making a purchase in each period is λ, which equals to q_K. On the other hand, the consumer has a reward r, redeemable in the next i periods if she made a purchase $K-i$ periods ago, but has not come to the market since then, the probability of which is $q_i = \lambda(1-\lambda)^{K-i}$. In particular, the consumer has no reward on hand if she does not come to the market in the previous K periods, the probability of which is $q_0 = (1-\lambda)^K$.

The reward r will only expire if the consumer does not come to the market for K consecutive periods, the probability of which is $(1-\lambda)^K$. It follows that the probability that the reward does not expire is $1-(1-\lambda)^K$. The term $r\left[1-(1-\lambda)^K\right]$ can be interpreted as the expected

[12]Since $h(i)$ increases in i, the utility difference between purchase and no-purchase increases as i decreases. This implies that the purchase option becomes more attractive relative to the no-purchase option as a consumer's reward draws closer to the expiration date. This result is consistent with the "point pressure" effect established in the literature; see, e.g., Kopalle et al. (2012). For analytical tractability, we treat consumers' purchase utility as deterministic. This is a limiting feature of our model, as it does not allow us to model purchase acceleration.

value of the reward, taking into account the probability of reward expiration. The condition $v + r\left[1-(1-\lambda)^K\right] \geq p$ simply states that a purchase is only made if the valuation plus the expected value of the reward exceeds the purchase price. Observe that the expected value of the reward is increasing in both r and K. Hence, to induce consumer purchases at a price p, the firm can raise either r or K, or both.

Breakage occurs when a consumer purchases the product and earns a reward, but does not come to the market for the subsequent K periods. The probability of reward expiration, $(1-\lambda)^K$, can be interpreted as the breakage rate. Frequent consumers purchase at higher frequencies and therefore incur lower breakage rates. Conditional on shopping probability, the breakage rate is lower for a larger K because consumers have more opportunities to redeem the reward. Therefore, the breakage rate is decreasing in both λ and K.

4 The Firm's Problem

In this section, we analyze the firm's revenue-maximization problem. We first analyze the case with a fixed K in Section 4.1. We then derive the optimal expiration term in Section 4.2. We build on this analysis to investigate consumer and social welfare in Section 4.3.

To write down the firm's long-run average revenue, we calculate the average revenue contribution of consumers who purchase in all states.[13] In each period, a consumer's revenue contributions for purchases with and without a reward are $p-r$ and p, respectively.[14] Furthermore, a consumer purchases without a reward only in state 0. Recall that q_i denotes the stationary probability of state i. The average revenue contribution of a consumer with shopping probability λ is given by

$$\lambda\left[q_0 p + (1-q_0)(p-r)\right] = \lambda\left[p - r\left(1-(1-\lambda)^K\right)\right].$$

The term $p - r\left(1-(1-\lambda)^K\right)$ is the *effective price* paid by the consumer. It is readily observed that the effective price increases with the regular price p, but decreases with the reward size r, and the expiration term K. Recall that there are four distinct consumer segments in the market. The firm's revenue is the sum of the revenue from the groups that purchase in all states. There are six possible market outcomes, depending on which subset of the four consumer segments purchases

[13] A consumer who does not purchase in every state will eventually exit the market and does not contribute to the firm's long-run average revenue.

[14] Here we consider the most inefficient reward (Kim et al., 2001), where the firm's profit is decreased by the same amount of the reward received by the consumer (e.g., price discount). Section 6.3 considers the situation where the firm can offer more efficient rewards.

from the firm.[15]

4.1 Analysis for a Fixed Expiration Term

We first analyze the problem for a fixed $K \geq 1$. Such an analysis provides the basis for subsequent analysis on the optimal K. Furthermore, it is relevant for situations where the expiration term is dictated by other considerations, such as competitive pressure or industry norms.

When K is fixed, the firm varies price p and reward r to maximize revenue. Lemma 1 shows that three are three possible market outcomes.

Lemma 1. *Suppose the expiration term is fixed at $K \geq 1$. There are three possible market outcomes at optimality:*

(i) ***Full market coverage without reward:*** *The optimal price, reward, and firm revenue are given by*

$$(p_1^*, r_1^*) = (v_L, 0),\ \ \pi_1^* = [\beta\lambda_F + (1-\beta)\lambda_I]v_L.$$

(ii) ***Partial market coverage without reward:*** *The optimal price, reward, and firm revenue are given by*

$$(p_2^*, r_2^*) = (v_H, 0), \pi_2^* = [\gamma\lambda_F + (\alpha-\gamma)\lambda_I]v_H.$$

(iii) ***Partial market coverage with reward:*** *The optimal price, reward, and firm revenue are given by*

$$\begin{aligned}
p_3^*(K) &= v_L + \left[1-(1-\lambda_F)^K\right] r_3^*, \\
r_3^*(K) &= \min\left\{\frac{v_H - v_L}{(1-\lambda_I)^K - (1-\lambda_F)^K}, \frac{v_L}{(1-\lambda_F)^K}\right\}, \\
\pi_3^*(K) &= \beta\lambda_F v_{\mathrm{L}} + (\alpha-\gamma)\lambda_I \min\left\{v_H, \left(\frac{1-\lambda_I}{1-\lambda_F}\right)^K v_L\right\}.
\end{aligned}$$

In market outcome (i), all consumers are served, the price is v_L, and no reward is offered. In market outcome (ii), only high valuation consumers are served, the price is v_H, and no reward is offered. Market outcome (iii) is an interesting case where a nontrivial reward is offered, and all but the low-valuation-infrequent consumers are served.

Lemma 1 suggests that offering a reward program can potentially increase the revenue of the firm, compared with not offering a reward program and charging either a high or low price. An

[15] Note that the purchase outcomes for consumer segments are ordered by valuation; when consumers with low valuations purchase, consumers with high valuations and the same shopping probability also purchase.

immediate question is when such a reward program is profitable; that is, when market outcome (iii) dominates outcomes (i) and (ii). Observe that, in practice, the reward program strategy can be dramatically different even for firms in the same industry. There is also substantial controversy regarding the effectiveness of reward programs in the industry. Indeed, several grocery retailers recently abandoned their established reward programs and reverted to "everyday low prices" (Tuttle, 2013). We show later that in our model, the optimal program adoption strategy depends on both types of consumer heterogeneity.

The next proposition states that a reward program is effective only when there is a relatively large segment of high-valuation-infrequent consumers. Otherwise, the firm is better off serving either all consumers or only high-valuation consumers and offering no reward.

Proposition 2. *For any expiration term $K \geq 1$, if $\gamma \geq \alpha\beta$, offering a reward program will not improve the firm's revenue.*

Proposition 2 states that offering a reward program will not improve the firm's revenue for *any* expiration term K if $\gamma \geq \alpha\beta$. Recall that when $\gamma \geq \alpha\beta$, the correlation coefficient, ρ, between valuation and shopping probability is positive. In other words, offering a reward program can only improve the firm's revenue when valuation and shopping probability are *negatively* correlated.

The result of Proposition 2 is a consequence of price discrimination under a reward program. In market outcome (iii), the effective price paid by frequent (both low- and high-valuation) consumers is $p_3^* - \left[1 - (1-\lambda_F)^K\right] r_3^* = v_L$. Therefore, when both price and reward are set optimally, frequent consumers pay an effective price of v_L, which ensures that all frequent consumers purchase in each state. Furthermore, the effective price paid by high-valuation-infrequent consumers is

$$\begin{aligned} p_3^* - \left[1 - (1-\lambda_I)^K\right] r_3^* &= v_L + \left[1 - (1-\lambda_F)^K\right] r_3^* - \left[1 - (1-\lambda_I)^K\right] r_3^* \\ &= \min\left\{ v_H, \left(\frac{1-\lambda_I}{1-\lambda_F}\right)^K v_L \right\}. \end{aligned} \tag{5}$$

The effective price paid by high-valuation-infrequent consumers (5) is between v_L and v_H. Since this effective price is above v_L, low-valuation-infrequent consumers do not purchase. However, the finite expiration term of a reward program enables the firm to exploit high-valuation-infrequent consumers with a higher effective price. Proposition 2 states that such a reward program is profitable only when high-valuation-infrequent segment is sufficiently large.

4.2 The Optimal Expiration Term

Lemma 1 provides a basis for analyzing the optimal expiration term decision. Since $\frac{1-\lambda_I}{1-\lambda_F} > 1$, the effective price paid by high-valuation-infrequent consumers (5) increases in K. Therefore, by adjusting K, the firm is able to extract more consumer surplus from these consumers. When K is large enough, the effective price paid by high-valuation-infrequent consumers becomes v_H. Since v_H is the highest effective price HI consumers are willing to pay, the firm cannot gain additional revenue by further increases in K.

Proposition 3. *When the reward program is offered, the optimal revenue is nondecreasing in K and is maximized at any $K \geq K^* = \left\lceil \frac{\log\left(\frac{v_H}{v_L}\right)}{\log\left(\frac{1-\lambda_I}{1-\lambda_F}\right)} \right\rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. For any K, the optimal price, reward, and revenue are given in Lemma 1(iii).*

Proposition 3 characterizes the optimal expiration term, which is not unique. For convenience, we refer to K^* as the optimal expiration term. For $K \geq K^*$, the effective price paid by frequent consumers is v_L and the effective price paid by high-valuation-infrequent consumers is v_H. Increasing the expiration term beyond K^* does not improve the firm's revenue.

We would like to emphasize that despite the multiplicity of optimal expiration terms, the firm may have practical considerations to set $K = K^*$. This is because as Proposition 5 below shows, the optimal price p^* increases with the expiration term. Thus, selecting a short K allows the firm to charge a low price. Additionally, it is imperative that the expiration term be finite. If rewards never expire, then rewards become *de facto* price reductions. Since the same rewards are offered to all consumers, they cannot be used to price discriminate among consumers anymore. Therefore, our model implies that a necessary condition for the effectiveness of reward programs is a finite expiration term. Indeed, most reward programs we observe in practice have finite expiration terms, and many companies actively modify the expiration terms of their reward programs. No-expiration policies can also be attributed to factors not considered in our model, such as the need to gain goodwill and achieve market differentiation. JetBlue initiated a huge advertisement campaign featuring its no-expiration policy.[16] Practices in the travel industry show that a no-expiration policy is often not permanent: for example, Marriott switched to a 24-month expiration policy starting on February 1, 2016.[17]

The optimal expiration term K^* increases in the ratio $\frac{v_H}{v_L}$, which can be viewed as a measure

[16] See, e.g., http://blog.jetblue.com/index.php/2013/06/18/trueblue-points-dont-expire-like-ever/. Retrieved on October 2, 2015.

[17] See http://www.marriott.com/marriott-rewards/terms/expiration.mi. Retrieved on October 24, 2015.

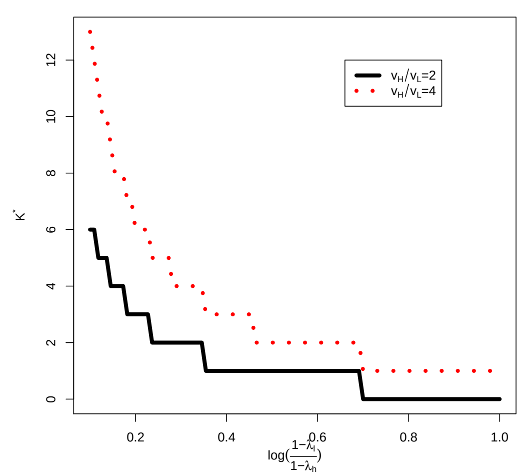
of consumer valuation difference. This implies that when valuation difference is higher (lower), the optimal expiration term is longer (shorter).

The value of K^* is decreasing in $\log(\frac{1-\lambda_I}{1-\lambda_F})$, which we refer to as *purchase frequency difference.* This implies that a smaller λ_F or a larger λ_I corresponds to longer expiration terms. Roughly speaking, if the shopping probabilities of frequent and infrequent consumers are closer, the optimal expiration term is longer. The solid and dotted lines in Figure 3 illustrate how K^* changes with respect to purchase frequency difference for $\frac{v_H}{v_L} = 2$ and 4, respectively.

Note that the breakage rate is determined by both the shopping probability and expiration term. Thus, Proposition 3 shows that the optimal breakage should be positive but not excessive. Without a finite expiration term, reward breakage becomes zero, and rewards become a de facto price reduction for all consumers, rendering the reward program ineffective. However, the optimal breakage rate should also be bounded by the level induced by the finite expiration term K^*.

Observation 1. *The optimal expiration term is longer (shorter) for retailers that have lower (higher) purchase frequency difference.*

Figure 3: Optimal expiration terms vs. purchase frequency difference



When the expiration term K is set optimally, when will each market outcome dominate? Recall that the result of Proposition 2 applies for any fixed K. Therefore, even if the expiration term is chosen optimally, a reward program cannot improve the firm's revenue if there is a positive correlation between valuation and shopping probability (i.e., high valuation consumers also tend to shop more frequently). Proposition 4 further delineates the conditions under which each market outcome is revenue-maximizing when K is chosen optimally.

Proposition 4. *With an optimal reward program denoted by (p^*, r^*, K^*), the optimal price, reward, and revenue are given by*

$$p^* = \begin{cases} v_L, & \text{if } \frac{v_H}{v_L} \leq \min\left\{\frac{1-\beta}{\alpha-\gamma}, \frac{\beta\lambda_F+(1-\beta)\lambda_I}{\gamma\lambda_F+(\alpha-\gamma)\lambda_I}\right\}, \\ v_H, & \text{if } \frac{v_H}{v_L} \geq \max\left\{\frac{\beta}{\gamma}, \frac{\beta\lambda_F+(1-\beta)\lambda_I}{\gamma\lambda_F+(\alpha-\gamma)\lambda_I}\right\}, \\ v_L + \left[1-(1-\lambda_F)^{K^*}\right] r^*, & \text{otherwise,} \end{cases}$$

$$r^* = \begin{cases} 0, & \text{if } \frac{v_H}{v_L} \leq \min\left\{\frac{1-\beta}{\alpha-\gamma}, \frac{\beta\lambda_F+(1-\beta)\lambda_I}{\gamma\lambda_F+(\alpha-\gamma)\lambda_I}\right\}, \\ 0, & \text{if } \frac{v_H}{v_L} \geq \max\left\{\frac{\beta}{\gamma}, \frac{\beta\lambda_F+(1-\beta)\lambda_I}{\gamma\lambda_F+(\alpha-\gamma)\lambda_I}\right\}, \\ \frac{v_H-v_L}{(1-\lambda_I)^{K^*}-(1-\lambda_F)^{K^*}}, & \text{otherwise,} \end{cases}$$

$$\pi^* = \begin{cases} \left[\beta\lambda_F+(1-\beta)\lambda_I\right] v_L, & \text{if } \frac{v_H}{v_L} \leq \min\left\{\frac{1-\beta}{\alpha-\gamma}, \frac{\beta\lambda_F+(1-\beta)\lambda_I}{\gamma\lambda_F+(\alpha-\gamma)\lambda_I}\right\}, \\ \left[\gamma\lambda_F+(\alpha-\gamma)\lambda_I\right] v_H, & \text{if } \frac{v_H}{v_L} \geq \max\left\{\frac{\beta}{\gamma}, \frac{\beta\lambda_F+(1-\beta)\lambda_I}{\gamma\lambda_F+(\alpha-\gamma)\lambda_I}\right\}, \\ \beta\lambda_F v_{\mathrm{L}} + (\alpha-\gamma)\lambda_I v_{\mathrm{H}}, & \text{otherwise.} \end{cases}$$

The third line in each of the above expressions describe the scenario where it is optimal for the firm to adopt a reward program. Note that both the optimal price (p^*) and optimal reward size (r^*) can be written as a function of the optimal expiration term, K^*. This consideration prompts us to investigate the relationships among p^*, r^*, and K^*.

One relationship of interest is that between the expiration term and reward size. Since the consumers' effective prices are lower when either the reward is larger or the expiration term is longer, we expect this relationship to be monotone, such that the reward size decreases with the expiration term. Intuitively, when K increases, consumers have more opportunities to redeem the rewards. Proposition 5 shows that this is not necessarily the case: the reward can increase or decrease with K, depending on market characteristics. In particular, when valuation difference is sufficiently high, the firm has an incentive to use both a larger reward *and* a longer expiration term to price discriminate between high- and low-valuation consumers.

Proposition 5. *When a reward program is offered with an optimal expiration term K^*,*

(i) the optimal reward $r_3^*(K^*)$ first decreases and then increases in K^*.

(ii) the optimal price $p_3^*(K^*)$ increases in K^* .

Two more empirically testable observations emanate from Proposition 5:

Observation 2. *The optimal expiration term is longer (shorter) for retailers that charge higher (lower) prices.*

Observation 3. *The relationship between the optimal expiration term and reward size is not monotone and moderated by the valuation and purchase frequency heterogeneity.*

It is also clear from Proposition 4 that the firm does not always benefit from the use of a reward program. Figures 4a–4c illustrate the optimal market outcomes when $\alpha = \beta = 0.5$. The three subfigures correspond to the three levels of correlation between valuation and shopping probability among the consumer population: 0 (no correlation), -0.5 (moderate and negative correlation), and -0.9 (high and negative correlation). The horizontal axis is $\frac{v_L}{v_H}$, and the vertical axis is $\frac{\lambda_I}{\lambda_F}$.[18] The two axes measure the *homogeneity* in valuation and shopping probability, respectively. Figure 4a shows that when the correlation is zero, the firm does not benefit from a reward program. Instead, the firm either serves the entire market with a low price (when valuation homogeneity $\frac{v_L}{v_H}$ is high) or only the high-valuation segments (when valuation homogeneity $\frac{v_L}{v_H}$ is low). This is consistent with Proposition 2. Figure 4b shows that when the correlation is moderately negative at -0.5, there is an intermediate range of valuation and shopping probability homogeneity, where all but low-valuation-infrequent consumers are served and a nontrivial reward is offered. Figure 4c shows that when the correlation becomes highly negative at -0.9, the firm is even more likely to offer a reward program.

Proposition 4 states that instead of using a reward program, the firm may use either a high or low price. This finding is consistent with the fact that many firms (e.g., Walmart and Whole Foods) make the deliberate decision not to have a reward program, using a relatively constant pricing strategy instead. However, across firms that do not offer reward programs, the equilibrium price levels may be very different: low (high) when the valuation homogeneity is high (low).

4.3 Impact on Consumer and Social Welfare

According to Lemma 1, when no reward is offered, the price paid by consumers is either v_L or v_H, depending on whether it is more profitable to serve all consumers or only high-valuation consumers. When a reward program with expiration term K is offered, the effective prices paid by frequent and infrequent consumers are, respectively, $p^* - \left[1 - (1-\lambda_F)^K\right] r^*$ and $p^* - \left[1 - (1-\lambda_I)^K\right] r^*$. Based on Part (iii) in Lemma 1, Table 1 summarizes the effective price paid by each consumer segment.

Based on Table 1, offering a reward program never hurts frequent consumers – HF consumers may pay a lower effective price under a reward program; LF consumers who are not served under market outcome (ii) are served when a reward program is offered. The story is different for

[18]We use v_L/v_H here, since it is strictly between 0 and 1, whereas v_H/v_L can take any value between 0 and infinity.

Figure 4: Market composition and optimal firm strategy

(a) $\rho = 0$ (b) ρ= -0.5 (c) ρ= -0.9

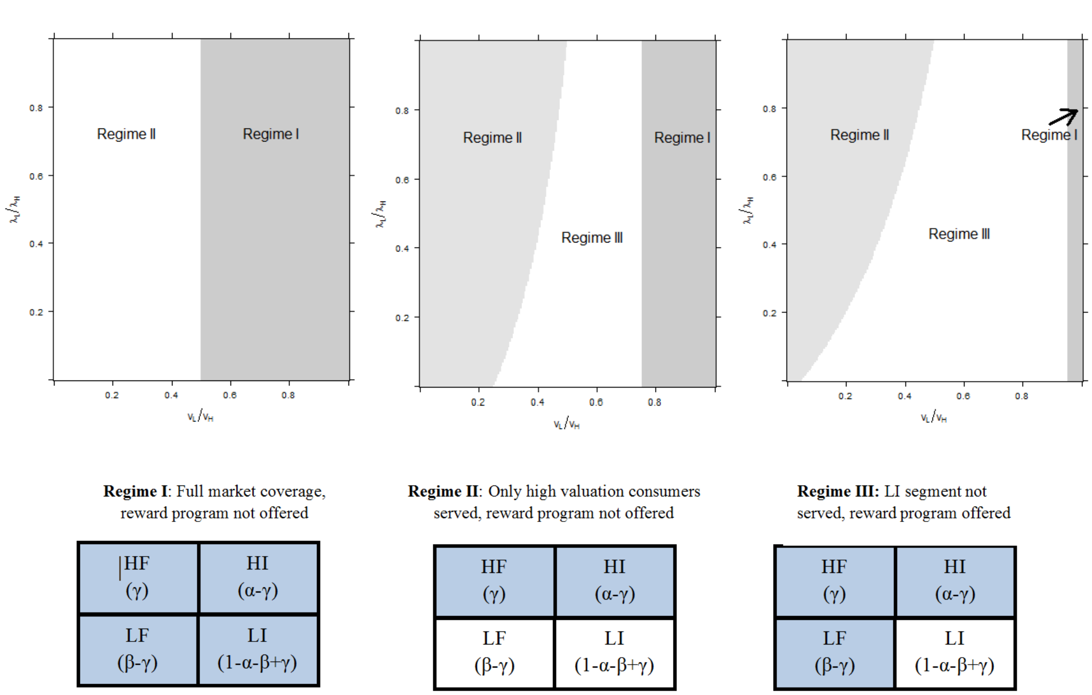


Table 1: Effective prices paid by each consumer segment under different market outcomes

Consumer Segment	Market Outcomes: (i) Full coverage, no reward	(ii) Partial coverage, no reward	(iii) Partial coverage, with reward
High-valuation-frequent	v_L	v_H	v_L
Low-valuation-frequent	v_L	Not served	v_L
High-valuation-infrequent	v_L	v_H	$\min\left\{v_H, \left(\frac{1-\lambda_I}{1-\lambda_F}\right)^K v_L\right\}$
Low-valuation-infrequent	v_L	Not served	Not served

infrequent consumers. When market outcome switches from (i) to (iii), HI consumers may end up paying a higher effective price and LI consumers are no longer served. In addition, the total consumer welfare decreases. In contrast, total consumer welfare increases when market outcome switches from (ii) to (iii). We summarize these observations in the following proposition.

Proposition 6. *For any $K \geq 1$, offering an optimal reward program with expiration term K (1) never hurts the consumer surplus of frequent consumers, and (2) can increase or decrease the overall consumer surplus of infrequent consumers, compared with the case without rewards. The total consumer welfare can be either higher or lower.*

What happens to social welfare, i.e., the sum of consumer welfare and the firm's revenue? Proposition 7 shows that the effect of a reward program on the social welfare is ambiguous as well, and echoes that on total consumer welfare.

Proposition 7. *For any $K \geq 1$, offering an optimal reward program with expiration term K can increase or decrease the overall social welfare, compared with the case without rewards.*

Propositions 6 and 7 highlight the role of valuation difference among consumers. When the valuation difference among consumers is high, it is optimal for the firm to only serve high-valuation consumers when a reward program is not offered; the reward program *increases* the total consumer/social welfare by attracting consumers who would not be served without the reward program. However, when the valuation difference among consumers is low, it is optimal for the firm to serve all consumers when a reward program is not offered; the reward program *decreases* the total consumer/social welfare by inducing the firm to exclude LI consumers.

5 Empirical Analysis

In this section, we show that the key predictions from our theoretical model are consistent with the finite reward expiration terms set by a broad range of retailers that offer reward programs. In particular, Proposition 3 establishes that the optimal expiration term K^* is inversely related to $\log\left(\frac{1-\lambda_I}{1-\lambda_F}\right)$, which we denote as Φ_λ for simplicity. Note that Φ_λ is greater than zero since $\lambda_F > \lambda_I$. Furthermore, Φ_λ can be interpreted as a measure for purchase frequency *heterogeneity* among consumers; for any given λ_F, Φ_λ decreases as λ_I approaches λ_F from below. Our theoretical predictions are summarized in Observations 1–3 in the previous section.

Data. We investigate the top 100 US retailers (based on total 2013 US sales) identified by the US National Retail Federation (NRF). Two sellers, AT&T and Verizon Wireless, are excluded from the analysis: both use a subscription model that is different from our model setup. The original NRF data[19] include neither information regarding the use of reward programs, nor consumers' purchase frequencies. We collect information on reward programs in two steps. In the first step, we visit each retailer's website to see if a reward program is offered, and if so, what the program details are, especially regarding the expiration policy. In the second step, we call the customer service number of each retailer to cross-validate information obtained in the previous step. We excluded "reward programs" that are *de facto* instant rebate programs (e.g., ShopRite), and those that require consumers to go beyond purchasing in order to earn rewards (e.g., Costco charges an extra $55 for "executive" membership that allows the consumer to earn 2% in cash rewards). In total, we identified 55 retailers that use reward programs. For each reward program, we documented the types of rewards offered (e.g., store credit, gas discount, etc.) and the expiration terms in months. Summary statistics of these variables are shown in the second panel of Table 3 in the Online Appendix.

We supplement the NRF dataset with the following additional firm characteristics: the number of employees, industry sector, and operating margin as of 2015. For publicly traded companies, we use a web crawler to collect this information from the *Company Profile* and *Key Statistics* sections of *Yahoo!* Finance. For private companies, we resort to other reliable information sources, including the company's own publications, Wikipedia, and Investopedia.

Since there is no publicly available information on consumer purchase frequencies, we conduct Mturk surveys with active consumers, defined as those who have shopped with the retailer at least once during the past 12 months. A&P, which declared bankruptcy in 2015, was excluded from the survey. Among the remaining 97 retailers, a majority (78) are national; however a significant number (19) are not.[20] For these non-national retailers, we use the IP addresses of the Mturk respondents as an additional screening criterion. Each confirmed active consumer is asked to recall the total number of times she has shopped at each retailer in the past year.[21] We divide this number by 52 to get the average weekly purchase frequency, which we denote by μ_λ. In the end, we obtained data from 2,500 consumers, averaging 125 responses for each retailer included in the survey. Table 3 in the Online Appendix below summarizes the purchase frequencies for these retailers.

We compute two retailer-specific measures for purchase frequency heterogeneity, which is the

[19] Accessed at https://nrf.com/2014/top100-table on February 10, 2017.

[20] For example, H-E-B operates in TX, Giant Eagle in PA, OH, WV, IN and MD, and ShopRite in six northeastern states.

[21] Each respondent has the option to indicate that she is unsure about or cannot exactly recall the number of shopping trips, in which case her response is dropped from further analyses.

basis of our focal theoretical prediction (Observation 1). The first measure is the standard deviation of the purchase frequencies (σ_λ) . The second measure is Φ_λ introduced earlier. To operationalize λ_I and λ_F, we first split consumers into high-frequency and low-frequency shoppers based on the median purchase frequency of all shoppers of the focal retailer. We then compute the average purchase frequencies of low- and high-frequency shoppers as an approximation for λ_I and λ_F, respectively. Results are summarized in the third panel of Table 3. We find that that these two measures are highly positively correlated with each other ($\rho_{\sigma_\lambda,\Phi_\lambda}$=0.86), and both are highly correlated with the mean purchase frequencies ($\rho_{\sigma_\lambda,\mu_\lambda}$=0.90 and $\rho_{\Phi_\lambda,\mu_\lambda}$=0.94).

Figures 5a and 5b in the Online Appendix plot the expiration term against each of the two measures of purchase frequency heterogeneity, and clearly show that the expiration terms are inversely related to either measure. Two univariate regressions find that these relationships are statistically significant ($\beta_{\sigma_\lambda} = -10.18, p$-value $= 0.037$; $\beta_{\Phi_\lambda} = -3.944, p$-value $= 0.039$).[22]

We next investigate the relationships of the expiration term to the purchase frequency heterogeneity, the operating margin, and the reward size, summarized in Observations 1–3. Our empirical analysis is based on the following specification:

$$\log(K) = \beta_0 + \beta_1 \cdot \sigma_\lambda + \beta_2 \cdot Margin + \beta_3 \cdot Reward + \beta_4 \cdot Reward \cdot \sigma_\lambda + \epsilon.$$

In the above equation, we use σ_λ as the measure for purchase frequency difference across firms. Column 1 of Table 2 reports the results. Observation 1 predicts a negative relationship between $\log(K)$ and σ_λ: this is supported by the parameter estimate ($\beta_1 = -1.54, p < 0.01$). The coefficient of operating margin is positive and significant, providing support for Observation 2 ($\beta_2 = 2.95, p < 0.01$). These results survive after we add the three firm-level control variables (Column 2 of Table 2), although the dip in adjusted R-squared suggests that these control variables did not add much to the explanatory power of the focal variables. Observation 3 suggests that the relationship between reward size and length of expiration term is ambiguous, and subject to the moderating effects of valuation and purchase frequency heterogeneity. Due to the difficulty of finding a proper measure of valuation heterogeneity, we only included the interaction with purchase frequency heterogeneity, i.e., $Reward \cdot \sigma_\lambda$. The coefficient for the reward size is significant and in the expected direction ($\beta_3 = -2.94, p < 0.05$), but the coefficient for the interaction term is positive but insignificant ($\beta_4 = 8.04, p > 0.10$). A caveat is that we may not have enough data to observe the whole spectrum

[22]We also repeated the analysis using the log transformation of expiration term, and find a similar negative relationships with the two measures of purchase frequency heterogeneity.

past the theoretical turning point. As a robustness check, we re-run the analysis using $\Phi(\lambda)$ as the alternative measure of purchase frequency difference. Columns 4 and 5 show the robustness of these results under this alternative specification.

Table 2: Estimation Results for Models on the Lengths of Expiration Terms

Models	**I**	**II**	**III**	**IV**	**V**	**VI**
Intercept	0.852***	0.819***	0.919***	0.699***	0.659***	0.767***
	(0.143)	(0.154)	(0.131)	(0.128)	(0.147)	(0.116)
Reward size	-2.944**	-2.988**	-2.22*	-1.939*	-1.926*	-1.459
	(1.332)	(1.408)	(1.278)	(1.104)	(1.149)	(1.051)
Operating margin	2.949**	3.122**		2.688**	2.512*	
	(1.210)	(1.483)		(1.230)	(1.480)	
σ_λ	-1.537***	-1.748***	-1.429***			
	(0.456)	(0.555)	(0.452)			
Reward$\cdot\sigma_\lambda$	8.039	9.058	5.075			
	(5.461)	(6.461)	(5.05)			
$\Phi(\lambda)$				-0.561**	-0.627**	-0.429**
				(0.207)	(0.257)	(0.194)
Reward$\cdot\Phi(\lambda)$				2.454	2.317	0.578
				(2.559)	(3.087)	(2.447)
US sales in 2013		0.002	0.0005		-0.0018	-0.0015
		(0.007)	(0.005)		(0.0074)	(0.005)
Number of stores		-0.0018	0.046**		0.0099	0.044
		(0.037)	(0.018)		(0.0389)	(0.019)
Number of employees		0.0002	-0.00018		0.0009	-0.00002
		(0.0014)	(0.001)		(0.0015)	(0.001)
Adjusted R-Squared	0.32	0.26	0.30	0.28	0.22	0.26

Notes: *: significant at the .10 level; **: significant at the .05 level; ***: significant at the .01 level.

We note that the above regression analysis is limited by the small number of observations, since it excludes retailers whose operating margins are unavailable. For robustness of our analysis, we report results of a regression with more data points that does not control for margin, which allows us to test Observations 1 and 3 using all 47 observations. The results are summarized in Columns 3 and 6. We find that the results are qualitatively the same. Specifically, the coefficients of σ_λ and $\Phi(\lambda)$ are consistently negative.

Brief Summary. First, we find empirical support for an inverse relationship between the purchase frequencies difference and expiration term. Consistent with Observation 1, retailers whose consumers exhibit larger differences in their purchase frequencies tend to set more stringent expiration terms. Second, we find a significant positive association between the expiration term and the operating margin, but not between the expiration term and the reward size. Importantly, these patterns hold after controlling for differences in retailer types and retailer-specific purchase frequencies. An obvious caveat of these results is that they are based on a relatively small sample; however, they do provide directional support for our theoretical predictions.

6 Model Extensions

In this section, we consider several extensions that relax certain assumptions in our main analysis. We show that our main results hold with minor adjustments.

6.1 Rush Redemption

The main model assumes that the shopping probability remains constant regardless of consumers' state and in particular, when the reward is close to expiration. Psychological research in anticipated regret theory (e.g. Bell, 1982; Loomes and Sugden, 1982) and empirical findings in the context of coupons (Inman and McAlister, 1994) suggests the possibility that consumers may be more likely to visit the seller in order to "salvage" the expiring rewards. To account for this possibility, we consider a model extension where the shopping probability is increased when a consumer's reward is near expiration. This extension substantially complicates the analysis. Yet, we find that neither the consumers' nor the firm's problems are qualitatively changed, and the insights of our main analysis remain qualitatively intact.[23] To conserve space, we relegate the model formulation and analysis to Online Appendix B.

6.2 Customized Reward Program Based on Shopping Frequencies

Our main analysis suggests that a consumer with shopping probability λ bases her purchase decision on the effective price $p - \left[1 - (1-\lambda)^K\right] r$. Since consumers can have different shopping probabilities, it is possible for the firm to customize prices, rewards and expiration terms based on the consumer's shopping probability. Thus, we consider another model extension that approximates the common practice of offering "tiered" rewards, where the reward structure is different for frequent

[23]One criticism for the rush redemption model is that the increase in consumers' shopping probability is non-strategic. We also explored several scenarios where consumers make strategic decisions of coming to the market. This necessitates additional tradeoff for the consumers. For example, a consumer may incur a random transportation cost T in each period. Our model is consistent with the case where T follows a two-point distribution. She shops in a period when T is small and skips the visit when T is large.

and infrequent consumers.[24] Suppose the product can be offered at two price–reward–expiration triplets, (p_1, r_1, K_1) and (p_2, r_2, K_2) to all customers.[25] The analysis in Section 4 corresponds to the special case where $(p_1, r_1, K_1) = (p_2, r_2, K_2)$. Note that, since there are only two consumer shopping probabilities, offering more than two price-reward-expiration triplets cannot increase the firm's revenue.

Proposition 8. *Any customized reward programs based on shopping probabilities cannot improve firm profit compared with the optimal reward program without customization.*

This result is surprising and at odds with the widely observed practice of offering reward programs with different tiers. However, an important caveat to this result is that our model does not take into account psychological factors that can play an important role in consumers' decision making. For example, being recognized as an "elite member" may provide consumers with intangible (e.g., perceived status), as well as tangible benefits (Drèze and Nunes, 2009).

6.3 Efficient Rewards

Our main analysis assumes that the firm's cost of offering a reward r is the same as r; i.e., the reward is the most inefficient (Kim et al., 2001). In practice, firms may choose to offer efficient rewards at lower cost. For example, airline frequent-flier programs may use surplus capacity to provide free tickets when demand is low (Kim et al., 2004). Efficient rewards can be modeled by introducing a parameter $\phi \in [0, 1]$ such that it costs the firm ϕr to provide a reward r. It can be shown that as ϕ decreases (i.e., the reward becomes more efficient), the firm has greater incentive to offer the reward program. Since it costs less to provide consumer rewards, the firm earns higher revenue. Proposition 6 shows that a reward program never hurts frequent consumers, regardless of their valuations, while the welfare impact of a reward program is mixed for infrequent consumers. Since the firm is more likely to offer a nontrivial efficient reward, consumer welfare for frequent consumers will be higher. The welfare impact on infrequent consumers remains ambiguous.

6.4 Nontrivial Outside Option

Our main model assumes that the outside option has a value of 0. We consider an extension where the outside option has a non-trivial value $v_o \in [0, v_L]$. Conceptually, v_o can be perceived as the

[24] For example, Costco offers 2% cash back for its more frequent (executive) members, but no cash back for less frequent (gold star) members.

[25] Here we assume that the firm cannot distinguish among the different types of customers and offer the tiered rewards to all customers.

competition intensity: a larger v_o means that the consumer's outside option is more attractive. We show that in the presence of a nontrivial outside option, i) consumers are better off, ii) for a fixed expiration term, a lower reward is offered together with a lower price, and iii) the optimal expiration term is longer. It would be interesting to investigate whether the same conclusions hold in an explicit, dynamic game-theoretical model. We leave such an extension to future research. To conserve space, we relegate the model formulation and analysis to Online Appendix D.

7 Summary and Future Research

We examined the finite reward expiration term, a prevalent, yet little understood component of reward programs. We find that it is possible for a firm to strategically set the expiration term to retain low-valuation-frequent consumers with a lower effective price, but charge high-valuation-infrequent consumers higher effective prices. Thus, a reward program with a finite expiration term may allow a firm to price discriminate against less frequent shoppers in repeated purchases over time. This intertemporal aspect of reward programs makes them distinct from several well-studied price discrimination mechanisms, including instant or mail-in rebates, coupons, and quantity discounts.

Our results provide several novel and practical managerial insights. We show that the market composition is key to the economic viability of customer reward programs. We then provide additional insights on how reward expiration terms should be set, and offer an explanation for why products with a higher (lower) purchase frequency difference should have shorter (longer) reward expiration terms. Furthermore, we show how the optimal expiration term relates to the firm's decisions on price and reward size.

Importantly, we find that a reward program can be effective, even if the reward is purely based on past purchases. This finding should be encouraging for firms that are wary of the high costs associated with utilizing consumer-level data (e.g., Berman, 2006), and consumers' increasing concerns over privacy (e.g., Leenheer et al., 2007; Tuttle, 2013). It suffices for the firm to know the overall market composition, instead of willingness to pay at the individual level. Hence, the informational requirement of implementing a reward program is minimal, compared with targeted pricing and targeted coupons. The market composition, of course, can be dynamic and the firm may need to change the expiration terms accordingly. We also show that the effectiveness of a reward program does not depend on its ability to increase the shopping probability of consumers, which is again encouraging for firms concerned about reward programs that do little to increase members' shopping probabilities (e.g., Leenheer et al., 2007; Sharp and Sharp, 1997).

For model parsimony and analytical tractability, we maintained several assumptions that can be relaxed in future research. First, we use discrete types to capture consumer heterogeneity; however, our model can be readily generalized to an arbitrary number of consumer types. Second, we focus on modeling how the expiration term leads to the underutilization of rewards by consumers. Future research can incorporate alternative mechanisms of breakage as well as reward stockpiling; see Stourm et al. (2015) for an excellent example. Third, future research can incorporate additional features of reward programs such as minimum purchase requirements, and can model additional consumer heterogeneity such as purchase quantity (Chun and Ovchinnikov, 2015). It is possible that introducing a redemption threshold will alter customer purchase behavior. For example, a customer might spend more than planned to reach the threshold. Introducing a redemption threshold, however, will not nullify the effect of finite expiration terms since the purpose of the finite expiration term is to exploit the difference in shopping probabilities among customers, while the redemption threshold seeks to exploit the difference in the purchase amount for each shopping occasion. Fourth, we focus on a monopolistic firm, and consider the extension of a non-trivial outside option as an implicit way to model competition. Future research can investigate how to set reward expiration terms for competing firms or firms in a coalition reward program. Finally, future empirical studies can potentially leverage quasi-experiments and especially exogenous changes in expiration terms to understand possible reverse-causality effect of expiration terms on consumer purchase frequencies.

References

Aviv, Y., A. Pazgal. 2008. Optimal pricing of seasonal products in the presence of forward-looking consumers. *Manufacturing and Service Operations Management* **10**(3) 339–359.

Aviv, Y., M. Wei. 2014. Innovative dynamic pricing: The potential benefits of early-purchase reward programs. Working paper, Olin School of Business, Washington University St. Louis.

Bakhtiari, Ali, BPS Murthi, Erin Steffes. 2013. Evaluating the effect of affinity card programs on customer profitability using propensity score matching. *Journal of Interactive Marketing* **27**(2) 83–97.

Bell, David E. 1982. Regret in decision making under uncertainty. *Operations research* **30**(5) 961–981.

Berman, Barry. 2006. Developing an effective customer loyalty program. *California management review* **49**(1) 123.

Besbes, Omar, Ilan Lobel. 2015. Intertemporal price discrimination: Structure and computation of optimal policies. *Management Science* **61**(1) 92–110.

Chun, So Yeon, Dan Andrei Iancu, Nikolaos Trichakis. 2015. Points for peanuts or peanuts for points? dynamic management of a loyalty program. Working paper, Georgetown University.

Chun, So Yeon, Anton Ovchinnikov. 2015. Strategic consumers, revenue management, and the design of loyalty programs. Working paper, Georgetown University.

Deighton, John A. 2000. *Frequency programs in service industries*. Thousand Oaks, CA: Sage Publications.

Drèze, Xavier, Joseph C Nunes. 2009. Feeling superior: the impact of loyalty program structure on consumers' perceptions of status. *Journal of Consumer Research* **35**(6) 890–905.

Hartmann, Wesley R, V Brian Viard. 2008. Do frequency reward programs create switching costs? a dynamic structural analysis of demand in a reward program. *Quantitative Marketing and Economics* **6**(2) 109–137.

Inman, J Jeffrey, Leigh McAlister. 1994. Do coupon expiration dates affect consumer behavior? *Journal of Marketing Research* **31**(3) 423–428.

Kim, Byung-Do, Peter E Rossi. 1994. Purchase frequency, sample selection, and price sensitivity: The heavy-user bias. *Marketing Letters* **5**(1) 57–67.

Kim, Byung-Do, Mengze Shi, Kannan Srinivasan. 2001. Reward programs and tacit collusion. *Marketing Science* **20**(2) 99–120.

Kim, Byung-Do, Mengze Shi, Kannan Srinivasan. 2004. Managing capacity through reward programs. *Management Science* **50**(4) 503–520.

Kopalle, Praveen K, Scott A Neslin. 2003. The economic viability of frequency reward programs in a strategic competitive environment. *Review of Marketing Science* **1**(1) 1–39.

Kopalle, Praveen K, Yacheng Sun, Scott A Neslin, Baohong Sun, Vanitha Swaminathan. 2012. The joint sales impact of frequency reward and customer tier components of loyalty programs. *Marketing Science* **31**(2) 216–235.

Leenheer, Jorna, Harald J Van Heerde, Tammo HA Bijmolt, Ale Smidts. 2007. Do loyalty programs really enhance behavioral loyalty? an empirical analysis accounting for self-selecting members. *International Journal of Research in Marketing* **24**(1) 31–47.

Lewis, Michael. 2004. The influence of loyalty programs and short-term promotions on customer retention. *Journal of Marketing Research* **41**(3) 281–292.

Liu, Yuping. 2007. The long-term impact of loyalty programs on consumer purchase behavior and loyalty. *Journal of Marketing* **71**(4) 19–35.

Loomes, Graham, Robert Sugden. 1982. Regret theory: An alternative theory of rational choice under uncertainty. *The economic journal* **92**(368) 805–824.

Lu, Xinwei, Xuanming Su. 2015. Revenue management with loyalty programs. Working paper, Wharton School.

O'Brien, Louise, Charles Jones. 1995. Do rewards really create loyalty? *Long range planning* **28**(4) 130–130.

Ross, S. 1983. *Introduction to stochastic dynamic programming*. Academic Press.

Rust, Roland T, Valarie A Zeithaml, Katherine N Lemon. 2001. *Driving customer equity: How customer lifetime value is reshaping corporate strategy*. Simon and Schuster.

Sharp, Byron, Anne Sharp. 1997. Loyalty programs and their impact on repeat-purchase loyalty patterns. *International Journal of Research in Marketing* **14**(5) 473–486.

Shu, Suzanne B, Ayelet Gneezy. 2010. Procrastination of enjoyable experiences. *Journal of Marketing Research* **47**(5) 933–944.

Stourm, Valeria, Eric T Bradlow, Peter S Fader. 2015. Stockpiling points in linear loyalty programs. *Journal of Marketing Research* **52**(2) 253–267.

Su, X. 2007. Inter-temporal pricing with strategic customer behavior. *Management Science* **53**(5) 726–741.

Su, X., F. Zhang. 2008. Strategic customer behavior, commitment, and supply chain performance. *Management Science* **54**(10) 1759–1773.

Su, X., F. Zhang. 2009. On the value of commitment and availability guarantees when selling to strategic consumers. *Management Science* **55**(5) 713–726.

Su, Xuanming. 2010. Intertemporal pricing and consumer stockpiling. *Operations research* **58**(4-part-2) 1133–1147.

Tuttle, Brad. 2013. A disloyalty movement? supermarkets and customers drop loyalty card programs. *Time.com* .

Winer, Russell S. 2001. A framework for customer relationship management. *California management review* **43**(4) 89–105.

Yin, R., Y. Aviv, A. Pazgal, C. Tang. 2009. Optimal markdown pricing: Implications of inventory display formats in the presence of strategic customers. *Management Science* **55**(8) 1391–1408.

Zhang, Z John, Aradhna Krishna, Sanjay K Dhar. 2000. The optimal choice of promotional vehicles: front-loaded or rear-loaded incentives? *Management Science* **46**(3) 348–362.

Online Appendix A: Proofs and Additional Analysis

Lemma 2 and Proof

Lemma 2. *Suppose $h(K) - h(0) \geq p - v$, then*

$$h(i) = h(0) + \lambda r \sum_{j=0}^{i-1} (1-\lambda)^j = h(0) + r\left[1 - (1-\lambda)^i\right], \qquad \forall i = 1, \ldots, K. \tag{6}$$

Consequently,

$$h(K) - h(i-1) = h(K) - h(0) - r\left[1 - (1-\lambda)^{i-1}\right] \geq p - v - r, \qquad \forall i = 1, \ldots, K.$$

Proof. The proof is by induction. We have

$$\begin{aligned} h(1) + g^* &= \lambda \max\{v - p + r + h(K), h(0)\} + (1-\lambda)h(0) \\ &= \lambda[v - p + r + h(K)] + (1-\lambda)h(0). \end{aligned} \tag{7}$$

In the above, the second equality uses $h(K) - h(0) \geq p - v$. We also have

$$h(0) + g^* = \lambda[v - p + h(K)] + (1-\lambda)h(0). \tag{8}$$

Taking the difference between (7) and (8), we have

$$h(1) - h(0) = \lambda r.$$

Therefore, the lemma holds for $i = 1$.

For the inductive step, suppose the lemma holds for a given $i = 1, \ldots, K-1$. We have

$$\begin{aligned} h(i+1) + g^* &= \lambda \max\{v - p + r + h(K), h(i)\} + (1-\lambda)h(i) \\ &= \lambda[v - p + r + h(K)] + (1-\lambda)h(i) \\ &= \lambda[v - p + r + h(K)] + (1-\lambda)h(i-1) + (1-\lambda)[h(i) - h(i-1)] \\ &= h(i) + g^* + \lambda r(1-\lambda)^i \\ &= g^* + h(0) + \lambda r \sum_{j=0}^{i} (1-\lambda)^j. \end{aligned} \tag{9}$$

In the above, for the second equality, we used

$$h(K) - h(i) = h(K) - h(0) - r\left[1 - (1-\lambda)^i\right] \geq p - v - r\left[1 - (1-\lambda)^i\right] \geq p - v - r.$$

Canceling g^*term on both sides in equation (9), we obtain

$$h(i+1) = h(0) + \lambda r \sum_{j=0}^{i} (1-\lambda)^j.$$

This completes the proof. $\square$

Proof of Proposition 1

The first half follows directly from Lemma 2.

Let q_i denote the stationary probabilities for state i, then q_i's satisfy the equations

$$\begin{aligned}
q_0 &= (1-\lambda)(q_0 + q_1), \\
q_i &= (1-\lambda) q_{i+1}, \qquad\qquad \forall i = 1, \ldots, K-1, \\
q_K &= \lambda \sum_{i=0}^{K} q_i, \\
\sum_{i=0}^{K} q_i &= 1.
\end{aligned}$$

Solving the above equations, we obtain Equation (4). $\square$

Proof of Lemma 1

Given (p, r), a consumer with valuation v and shopping probability λ purchases the product only if her valuation v is higher than the effective price $p - \left[1 - (1-\lambda)^K\right] r$. There are four distinct groups of consumers. Since $v_H \geq v_L$ and $p - \left[1 - (1-\lambda_I)^K\right] r \geq p - \left[1 - (1-\lambda_F)^K\right] r$, we need to discuss six cases.

Case 1: $p - \left[1 - (1-\lambda_F)^K\right] r \leq p - \left[1 - (1-\lambda_I)^K\right] r \leq v_L \leq v_H$

In this case, all consumers purchase in all states. The firm's revenue is given by

$$\begin{aligned}
&\beta\lambda_F\left(p-\left[1-(1-\lambda_F)^K\right]r\right)+(1-\beta)\lambda_I\left(p-\left[1-(1-\lambda_I)^K\right]r\right)\\
&\qquad=\left[\beta\lambda_F+(1-\beta)\lambda_I\right]p-\left(\beta\lambda_F\left[1-(1-\lambda_F)^K\right]+(1-\beta)\lambda_I\left[1-(1-\lambda_I)^K\right]\right)r.
\end{aligned}$$

The firm's profit-maximization problem can be formulated as

$$\begin{aligned}
&\max_{p\geq r\geq 0}\left[\beta\lambda_F+(1-\beta)\lambda_I\right]p-\left(\beta\lambda_F\left[1-(1-\lambda_F)^K\right]+(1-\beta)\lambda_I\left[1-(1-\lambda_I)^K\right]\right)r\\
&p-\left[1-(1-\lambda_I)^K\right]r\leq v_L.
\end{aligned}$$

The optimal solution is given by $(p^*,r^*)=(v_L,0)$ with corresponding revenue $\pi_1^*=\left[\beta\lambda_F+(1-\beta)\lambda_I\right]v_L$.

Case 2: $v_L\leq p-\left[1-(1-\lambda_F)^K\right]r\leq p-\left[1-(1-\lambda_I)^K\right]r\leq v_H$

In this case, only consumers with valuation v_H purchase in all states. The firm's profit maximization problem is given by

$$\begin{aligned}
&\max_{p\geq r\geq 0}\gamma\lambda_F\left(p-\left[1-(1-\lambda_F)^K\right]r\right)+(\alpha-\gamma)\lambda_I\left(p-\left[1-(1-\lambda_I)^K\right]r\right)\\
&p-\left[1-(1-\lambda_F)^K\right]r\geq v_L,\quad p-\left[1-(1-\lambda_I)^K\right]r\leq v_H.
\end{aligned}$$

The optimal solution is given by $(p^*,r^*)=(v_H,0)$. The corresponding revenue is $\pi_2^*=\left[\gamma\lambda_F+(\alpha-\gamma)\lambda_I\right]v_H$.

Case 3: $p-\left[1-(1-\lambda_F)^K\right]r\leq v_L\leq p-\left[1-(1-\lambda_I)^K\right]r\leq v_H$

In this case, only consumers with valuation v_L and shopping probability λ_I do not purchase in all states. The firm's revenue is given by

$$\begin{aligned}
&\beta\lambda_F\left(p-\left[1-(1-\lambda_F)^K\right]r\right)+(\alpha-\gamma)\lambda_I\left(p-\left[1-(1-\lambda_I)^K\right]r\right)\\
&\qquad=\left[\beta\lambda_F+(\alpha-\gamma)\lambda_I\right]p-\left(\beta\lambda_F\left[1-(1-\lambda_F)^K\right]+(\alpha-\gamma)\lambda_I\left[1-(1-\lambda_I)^K\right]\right)r.
\end{aligned}$$

The firm's profit-maximization problem can be formulated as

$$\begin{aligned}
&\max_{p\geq r\geq 0}\left[\beta\lambda_F+(\alpha-\gamma)\lambda_I\right]p-\left(\beta\lambda_F\left[1-(1-\lambda_F)^K\right]+(\alpha-\gamma)\lambda_I\left[1-(1-\lambda_I)^K\right]\right)r\\
&\qquad p-\left[1-(1-\lambda_F)^K\right]r\leq v_L\leq p-\left[1-(1-\lambda_I)^K\right]r\leq v_H.
\end{aligned}$$

The constraints can be rewritten as

$$p \le v_L + \left[1 - (1-\lambda_F)^K\right] r, \quad p \ge v_L + \left[1 - (1-\lambda_I)^K\right] r, \quad p \le v_H + \left[1 - (1-\lambda_I)^K\right] r.$$

Since the objective function increases in p, we must have

$$p = \min\left\{ v_L + \left[1 - (1-\lambda_F)^K\right] r, \quad v_H + \left[1 - (1-\lambda_I)^K\right] r \right\}.$$

If $v_L + \left[1 - (1-\lambda_F)^K\right] r \le v_H + \left[1 - (1-\lambda_I)^K\right] r$, we have $p = v_L + \left[1 - (1-\lambda_F)^K\right] r$ and $r \le \frac{v_H - v_L}{(1-\lambda_I)^K - (1-\lambda_F)^K}$. The constraint $p \ge r$ leads to $p = v_L + \left[1 - (1-\lambda_F)^K\right] r \ge r$, which gives $r \le \frac{v_L}{(1-\lambda_F)^K}$. The profit-maximization problem can be rewritten as

$$\max_{r \ge 0} \left[\beta\lambda_F + (\alpha-\gamma)\lambda_I\right] \left(v_L + \left[1 - (1-\lambda_F)^K\right] r \right) - \left(\beta\lambda_F \left[1 - (1-\lambda_F)^K\right] + (\alpha-\gamma)\lambda_I \left[1 - (1-\lambda_I)^K\right] \right) r$$

$$r \le \frac{v_H - v_L}{(1-\lambda_I)^K - (1-\lambda_F)^K}, \quad r \le \frac{v_L}{(1-\lambda_F)^K}.$$

The coefficient for r above is non-negative. Therefore, at optimality, we have

$$\hat{r} = \min\left\{ \frac{v_H - v_L}{(1-\lambda_I)^K - (1-\lambda_F)^K}, \frac{v_L}{(1-\lambda_F)^K} \right\}, \quad \hat{p} = v_L + \left[1 - (1-\lambda_F)^K\right] \hat{r}.$$

If $v_L + \left[1 - (1-\lambda_F)^K\right] r \ge v_H + \left[1 - (1-\lambda_I)^K\right] r$, we have $p = v_H + \left[1 - (1-\lambda_I)^K\right] r$ and $r \le \frac{v_H - v_L}{(1-\lambda_I)^K - (1-\lambda_F)^K}$. The constraint $p \ge r$ leads to $p = v_H + \left[1 - (1-\lambda_I)^K\right] r \ge r$, which gives $r \le \frac{v_L}{(1-\lambda_F)^K}$. Therefore, this case is only valid when $\frac{v_H - v_L}{(1-\lambda_I)^K - (1-\lambda_F)^K} \le \frac{v_L}{(1-\lambda_F)^K}$. The profit-maximization problem can be rewritten as

$$\max_{r \ge 0} \left[\beta\lambda_F + (\alpha-\gamma)\lambda_I\right] \left(v_H + \left[1 - (1-\lambda_I)^K\right] r \right) - \left(\beta\lambda_F \left[1 - (1-\lambda_F)^K\right] + (\alpha-\gamma)\lambda_I \left[1 - (1-\lambda_I)^K\right] \right) r$$

$$\frac{v_H - v_L}{(1-\lambda_I)^K - (1-\lambda_F)^K} \le r \le \frac{v_L}{(1-\lambda_I)^K}.$$

After some simplification, it can be shown that the objective function decreases in r. Hence, at optimality,

$$\tilde{r} = \frac{v_H - v_L}{(1-\lambda_I)^K - (1-\lambda_F)^K}, \quad \tilde{p} = v_H + \lambda_F \tilde{r}.$$

Observe that when $r = \frac{v_H - v_L}{(1-\lambda_I)^K - (1-\lambda_F)^K}$, we have $v_H + \left[1 - (1-\lambda_I)^K\right] r = v_L + \left[1 - (1-\lambda_F)^K\right] r$.

Therefore, the case $v_L + \left[1-(1-\lambda_F)^K\right] r \geq v_H + \left[1-(1-\lambda_I)^K\right] r$ is weakly dominated by the first case. We conclude that the optimal solution is given by

$$r^* = \min\left\{\frac{v_H - v_L}{(1-\lambda_I)^K - (1-\lambda_F)^K}, \frac{v_L}{(1-\lambda_F)^K}\right\},\ p^* = v_L + \left[1-(1-\lambda_F)^K\right] r^*.$$

The corresponding revenue is

$$\pi_3^* = \begin{cases} \beta\lambda_F v_{\mathrm{L}} + (\alpha-\gamma)\lambda_I v_{\mathrm{H}}, & \text{if } v_H \leq \frac{(1-\lambda_I)^K v_L}{(1-\lambda_F)^K}, \\ \beta\lambda_F v_{\mathrm{L}} + (\alpha-\gamma)\lambda_I v_{\mathrm{L}} \left(\frac{1-\lambda_I}{1-\lambda_F}\right)^K, & \text{if } v_H > \frac{(1-\lambda_I)^K v_L}{(1-\lambda_F)^K}. \end{cases}$$

$$= \beta\lambda_F v_{\mathrm{L}} + (\alpha-\gamma)\lambda_I \min\left\{v_H, \left(\frac{1-\lambda_I}{1-\lambda_F}\right)^K v_L\right\}.$$

Case 4: $p - \left[1-(1-\lambda_F)^K\right] r \leq v_L \leq v_H \leq p - \left[1-(1-\lambda_I)^K\right] r$

In this case, only consumers with shopping probability λ_F purchase in all states. The firm's profit-maximization problem is given by

$$\max_{p\geq r\geq 0} \beta\lambda_F \left(p - \left[1-(1-\lambda_F)^K\right] r\right)$$
$$p - \left[1-(1-\lambda_F)^K\right] r \leq v_L, p - \left[1-(1-\lambda_I)^K\right] r \geq v_H.$$

Note that the optimal revenue is bounded above by $\beta\lambda_F v_{\mathrm{L}}$; hence this case is dominated by Case 1, and will never be optimal.

Case 5: $p - \left[1-(1-\lambda_F)^K\right] r \leq v_L \leq p - \left[1-(1-\lambda_I)^K\right] r \leq v_H$

In this case, only consumers with valuation v_H and shopping probability λ_F purchase in all states. The firm's profit-maximization problem is given by

$$\max_{p\geq r\geq 0} \gamma\lambda_F \left(p - \left[1-(1-\lambda_F)^K\right] r\right)$$
$$p - \left[1-(1-\lambda_F)^K\right] r \leq v_L \leq p - \left[1-(1-\lambda_I)^K\right] r \leq v_H.$$

The optimal revenue is $\gamma\lambda_F v_{\mathrm{H}}$. Hence, this case is dominated by Case 3 and will never be optimal.

Case 6: $v_L \leq v_H \leq p - \left[1-(1-\lambda_F)^K\right] r \leq p - \left[1-(1-\lambda_I)^K\right] r$

In this case, no consumers purchase in either state, and the optimal revenue is 0. Obviously, this case will never be optimal. □

Proof of Proposition 2

It suffices to show that the revenue in Case (iii) is dominated by that of Cases (i) and (ii) in Lemma 1. We have

$$\begin{aligned}\pi_3^* - \pi_1^* &= (\alpha-\gamma)\lambda_I \min\left\{v_H, \left(\frac{1-\lambda_I}{1-\lambda_F}\right)^K v_L\right\} - (1-\beta)\lambda_I v_{\mathrm{L}} \\ &\le (\alpha-\gamma)\lambda_I v_{\mathrm{H}} - (1-\beta)\lambda_I v_{\mathrm{L}} \\ &\le (1-\beta)\lambda_I(\alpha v_H - v_L).\end{aligned}$$

In the second inequality above, we used $\gamma \ge \alpha\beta$.

We also have

$$\begin{aligned}\pi_3^* - \pi_2^* &= \beta\lambda_F v_{\mathrm{L}} + (\alpha-\gamma)\lambda_I \min\left\{v_H, \left(\frac{1-\lambda_I}{1-\lambda_F}\right)^K v_L\right\} - [\gamma\lambda_F + (\alpha-\gamma)\lambda_I]v_H \\ &= \lambda_F(\beta v_L - \gamma v_H) + (\alpha-\gamma)\lambda_I \min\left\{0, \left(\frac{1-\lambda_I}{1-\lambda_F}\right)^K v_L - v_H\right\} \\ &\le \lambda_F(\beta v_L - \gamma v_H) \\ &\le \lambda_F\beta(v_L - \alpha v_H).\end{aligned}$$

Therefore, $\pi_3^* \le \pi_1^*$ when $\alpha v_H \le v_L$ and $\pi_3^* \le \pi_2^*$ otherwise. This completes the proof. □

Proof of Proposition 3

From Part (iii) of Lemma 1, $\pi_3^*(K)$ is non-decreasing in K and stays constant when

$$\left(\frac{1-\lambda_I}{1-\lambda_F}\right)^K \ge \frac{v_H}{v_L}. \tag{10}$$

Let K^* be the smallest integer such that (10) holds. Then K^* takes the form given in the proposition. Furthermore, for any $K \ge K^*$, the revenue is also maximized. This completes the proof.

□

Proof of Proposition 4

First, it can be shown that market outcome (i) dominates market outcome (ii) when $\frac{v_H}{v_L} \le \frac{\beta\lambda_F+(1-\beta)\lambda_I}{\gamma\lambda_F+(\alpha-\gamma)\lambda_I}$. Therefore, we need to compare $\pi_3^\dagger$ with π_1^* when $\frac{v_H}{v_L} \le \frac{\beta\lambda_F+(1-\beta)\lambda_I}{\gamma\lambda_F+(\alpha-\gamma)\lambda_I}$ and compare

$\pi_3^\dagger$ with π_2^* otherwise.

Suppose $\frac{v_H}{v_L} \leq \frac{\beta\lambda_F+(1-\beta)\lambda_I}{\gamma\lambda_F+(\alpha-\gamma)\lambda_I}$. We have $\pi_3^\dagger - \pi_1^* = (\alpha-\gamma)\lambda_I v_{\mathrm{H}} - (1-\beta)\lambda_I v_{\mathrm{L}}$. The difference is positive if $\frac{v_H}{v_L} > \frac{1-\beta}{\alpha-\gamma}$.

Now suppose $\frac{v_H}{v_L} > \frac{\beta\lambda_F+(1-\beta)\lambda_I}{\gamma\lambda_F+(\alpha-\gamma)\lambda_I}$. We have $\pi_3^\dagger - \pi_2^* = \beta\lambda_F v_{\mathrm{L}} - \gamma\lambda_F v_{\mathrm{H}}$. The difference is positive if $\frac{v_H}{v_L} < \frac{\beta}{\gamma}$.

Summarizing the analysis above and using Proposition 3 lead to the desired result. $\square$

Proof of Proposition 5

(i) Take the derivative of $r^*$with respect to K^*:

$$\frac{dr^*}{dK^*} = \frac{(v_H - v_L)\left[(1-\lambda_F)^{K^*}\log(1-\lambda_F) - (1-\lambda_I)^{K^*}\log(1-\lambda_I)\right]}{\left[(1-\lambda_F)^{K^*} - (1-\lambda_I)^{K^*}\right]^2},$$

which is less (greater) than zero when $K^*$is less (greater) than$-\frac{\log\left[\frac{\log(1-\lambda_F)}{\log(1-\lambda_I)}\right]}{\log(1-\lambda_F)-\log(1-\lambda_I)}$. Thus, the optimal reward size, r^*, first decreases and then increases with K^*; i.e., the relationship between $r^*$and $K^*$is non-monotonic. (ii) Take the derivative of p^* with respect to K^*gives

$$\frac{dp^*}{dK^*} = \frac{(v_H - v_L)\left[\left((1-\lambda_F)^{K^*} - 1\right)(1-\lambda_I)^{K^*}\log(1-\lambda_I) - (1-\lambda_F)^{K^*}\left((1-\lambda_I)^{K^*} - 1\right)\log(1-\lambda_F)\right]}{\left[(1-\lambda_F)^{K^*} - (1-\lambda_I)^{K^*}\right]^2}$$

We would like to show that $\frac{dp^*}{dK^*}$ is non-negative for all positive K^*.

Since $(v_H - v_L) > 0$ and $\left[\left((1-\lambda_F)^{K^*} - (1-\lambda_I)^{K^*}\right)\right]^2 > 0$, the sign of $\frac{dp^*}{dK^*}$ is determined by the term in the square bracket in the numerator. To show that $\frac{dp^*}{dK^*} \geqslant 0$, it suffices to show that $\left((1-\lambda_F)^{K^*} - 1\right)(1-\lambda_I)^{K^*}\log(1-\lambda_I) - (1-\lambda_F)^{K^*}\left((1-\lambda_I)^{K^*} - 1\right)\log(1-\lambda_F) \geqslant 0$, or equivalently,

$$\frac{(1-\lambda_I)^{K^*}\log(1-\lambda_I)}{(1-\lambda_I)^{K^*} - 1} \geqslant \frac{(1-\lambda_F)^{K^*}\log(1-\lambda_F)}{(1-\lambda_F)^{K^*} - 1},$$

Define $g(x, K^*) \equiv \frac{(1-x)^{K^*}\log(1-x)}{\left((1-x)^{K^*}-1\right)}$. It suffices to show that $g(x, K^*)$ is decreasing in x for all $K^* \geqslant 1$. We have

$$\frac{dg(x,K^*)}{dx} = -\frac{(1-x)^{K^*-1}\left((1-x)^{K^*} - K^*\log(1-x) - 1\right)}{\left[(1-x)^{K^*} - 1\right]^2}.$$

From the above expression, it is easy to see that $\frac{dg(x,K^*)}{dx} \leq 0$ if and only if $h(x, K^*) \equiv (1-$

$x)^{K^*} - K^* \log(1-x) - 1 \geqslant 0$ for $K^* \geqslant 1$,

Note that $h(x, K^*)$ increases in K^* for all $x \geqslant 0$, and $h(x, K^*)$ increases in x for all $K^* \geqslant 1$. Since $\frac{dh(x,K^*)}{dK^*} = \big((1-x)^K - 1\big)\log(1-x) \geqslant 0$ and $\frac{dh(x,K^*)}{dx} = \frac{K\left((1-x)^K-1\right)}{x-1} \geqslant 0$, it follows that $h(x, K^*) \geqslant h(0,1) = -0 - \log(1) = 0$.

Thus, the optimal price p^* is non-decreasing with respect to K^*. This completes the proof. $\square$

Proof of Proposition 6

Observe that all frequent consumers are served in market outcomes (i) and (iii). In market outcome (ii), only high-valuation consumers are served, whereas in market outcome (iii) only low-valuation-infrequent consumers are not served. The following table summarizes the welfare impact of switching from market outcome (i) to (iii) and (ii) to (iii).

Impact on consumer welfare	HF	LF	HI	LI
(i) to (iii)	none	none	$-$	$(-)$
(ii) to (iii)	$+$	$(+)$	$+$	none

In the table above, we use $(+)$ to denote weak dominance when consumers switch from no-purchase to purchase with zero surplus. Similarly, we use $(-)$ to denote weak dominance when consumers switch from purchase with zero surplus to no-purchase. When switching from (i) to (iii), frequent consumers pay the same effective price. HI consumers pay a higher price, while LI consumers are priced out of the market.

When switching from (ii) to (iii), HF consumers pay a lower price. LF consumers are not served in outcome (ii), but are served in outcome (iii) with zero surplus. HI consumers pay a lower effective price in outcome (iii) than in (ii). LI consumers are not served in either (ii) or (iii).

The total consumer welfare for market outcomes (i), (ii), and (iii), are, respectively

$$CW_1 = [\gamma\lambda_F + (\alpha-\gamma)\lambda_I](v_H - v_L),$$
$$CW_2 = 0,$$
$$CW_3 = \gamma\lambda_F(v_H - v_L) + (\alpha-\gamma)\lambda_I\left(v_H - \left(\frac{1-\lambda_I}{1-\lambda_F}\right)^K v_L\right)^+.$$

Therefore, the total consumer welfare of outcome (iii) is lower than that of outcome (i) but higher than that of outcome (ii). $\square$

Proof of Proposition 7

The total social welfare (SW) for market outcomes (i), (ii), and (iii), are, respectively,

$$SW_1 = [\gamma\lambda_F + (\alpha-\gamma)\lambda_I]v_H + [(\beta-\gamma)\lambda_F + (1-\alpha-\beta+\gamma)\lambda_I]v_L,$$
$$SW_2 = [\gamma\lambda_F + (\alpha-\gamma)\lambda_I]v_H,$$
$$SW_3 = [\gamma\lambda_F + (\alpha-\gamma)\lambda_I]v_H + (\beta-\gamma)\lambda_F v_{\mathrm{L}}.$$

Therefore, the total social welfare of outcome (iii) is lower than that of outcome (i) but higher than that of outcome (ii). $\square$

Proof of Proposition 8

Without loss of generality, assume frequent consumers prefer (p_1, r_1, K_1) and infrequent consumers prefer (p_2, r_2, K_2). This assumption implies that

$$p_1 - \left[1-(1-\lambda_F)^{K_1}\right] r_1 \le p_2 - \left[1-(1-\lambda_F)^{K_2}\right] r_2 \le p_2 - \left[1-(1-\lambda_I)^{K_2}\right] r_2 \le p_1 - \left[1-(1-\lambda_I)^{K_1}\right] r_1.$$

In the above, the second inequality follows from $\lambda_F \ge \lambda_I$.

The firm's revenue depends on which groups purchase in all states in each period. We discuss six possible cases. Note that these six cases cover all possible scenarios.

<u>Case 1: $p_1 - \left[1-(1-\lambda_F)^{K_1}\right] r_1 \le p_2 - \left[1-(1-\lambda_I)^{K_2}\right] r_2 \le v_L \le v_H$</u>

In this case, all consumers purchase in all states. Recall that we assume frequent consumers prefer (p_1, r_1) and infrequent consumers prefer (p_2, r_2). The firm's profit-maximization problem can be formulated as

$$\max_{p_1\ge r_1\ge 0, p_2\ge r_2\ge 0, K_1\ge 1, K_2\ge 1} \beta\lambda_F\left(p_1 - \left[1-(1-\lambda_F)^{K_1}\right] r_1\right) + (1-\beta)\lambda_I\left(p_2 - \left[1-(1-\lambda_I)^{K_2}\right] r_2\right)$$
$$p_1 - \left[1-(1-\lambda_F)^{K_1}\right] r_1 \le p_2 - \left[1-(1-\lambda_I)^{K_2}\right] r_2 \le v_L.$$

Obviously, the objective value is bounded by $(\beta\lambda_F + (1-\beta)\lambda_I)v_L$, which is the revenue from offering price v_L and no reward. Hence, the revenue in this case will not exceed offering a common price and no reward.

<u>Case 2: $p_1 - \left[1-(1-\lambda_F)^{K_1}\right] r_1 \le v_L \le p_2 - \lambda_I r_2 \le v_H$</u>

In this case, all frequent consumers purchase at price p_1, while only infrequent consumers with valuation v_H purchase at price p_2. The firm's profit-maximization problem can be formulated as

$$\max_{p_1 \geq r_1 \geq 0, p_2 \geq r_2 \geq 0, K_1 \geq 1, K_2 \geq 1} \beta\lambda_F \left(p_1 - \left[1 - (1-\lambda_F)^{K_1}\right] r_1\right) + (\alpha - \gamma)\lambda_I \left(p_2 - \left[1 - (1-\lambda_I)^{K_2}\right] r_2\right)$$
$$p_1 - \left[1 - (1-\lambda_F)^{K_1}\right] r_1 \leq v_L \leq p_2 - \left[1 - (1-\lambda_I)^{K_2}\right] r_2 \leq v_H.$$

The corresponding optimal revenue is bounded by $\beta\lambda_F v_{\mathrm{L}} + (\alpha - \gamma)\lambda_I v_{\mathrm{H}}$. This revenue can be achieved by optimally chosen prices, rewards, and expiration terms without customization. Therefore, it is achievable.

Case 3: $v_L \leq p_1 - \lambda_F r_1 \leq p_2 - \lambda_I r_2 \leq v_H$

In this case, consumers with valuation v_L do not purchase; frequent consumers with valuation v_H purchase at price p_1, while infrequent consumers with valuation v_H purchase at price p_2. Obviously, the optimal solution is to offer price $p_1 = p_2 = v_H$ and $r_1 = r_2 = 0$.

Case 4: $p_1 - \lambda_F r_1 \leq v_L \leq v_H \leq p_2 - \lambda_I r_2$

In this case, all frequent consumers purchase at price p_1, while infrequent consumers do not purchase. Note that segmentation by shopping probability is completely ineffective in this case.

Case 5: $v_L \leq p_1 - \lambda_F r_1 \leq v_H \leq p_2 - \lambda_I r_2$

In this case, only frequent consumers with valuation v_H purchase at price p_1, while all other consumers do not purchase. This case is dominated by Case 3.

Case 5: $v_L \leq v_H \leq p_1 - \lambda_F r_1 \leq p_2 - \lambda_I r_2$

In this case, no consumers purchases. Clearly, this case cannot be optimal. □

Online Appendix B: A Model with Rush Redemption

Our main analysis assumes constant shopping probability for each customer. In particular, a customer's shopping probability does not change even when the reward is near expiration. Psychological research in in anticipated regret theory (e.g., Bell (1982); Loomes and Sugden (1982)) and empirical findings in the context of coupons (Inman and McAlister, 1994) suggests the possibility that the consumer may be more likely to visit the seller in order to "salvage" the expiring reward. In this section, we consider an extension where the shopping probability increases when a customer's reward is about to expire. Specifically, for a customer with baseline shopping probability λ, the shopping probability increases to $\bar{\lambda} \geq \lambda$ when the reward is about to expire (state 1), where $\bar{\lambda} \leq 1$. The dynamic programming equations change to

$$g^* + h(i) = \begin{cases} \lambda \max\{v - p + h(K), h(0)\} + (1-\lambda)h(0), & \text{if } i = 0, \\ \bar{\lambda} \max\{v - p + r + h(K), h(0)\} + (1-\bar{\lambda})h(0), & \text{if } i = 1, \\ \lambda \max\{v - p + r + h(K), h(i-1)\} + (1-\lambda)h(i-1), & \text{if } i = 2, \dots, K, \end{cases} \quad \forall i = 0, 1, \dots, K. \tag{11}$$

Proposition 9. *Suppose*

$$[1 - (1-\lambda)^{K-1} + (1-\lambda)^{K-1}\bar{\lambda}]r + v - p \geq 0, \tag{12}$$

then a solution to the optimality equations (11) is given by

$$h(0) = 0,$$

$$h(i) = (1-\lambda)^{i-1}[(\bar{\lambda} - \lambda)(v - p + r + h(K)) + \lambda r] + (1 - (1-\lambda)^{i-1})r, \qquad \forall i = 1, \dots, K-1, \tag{13}$$

$$h(K) = \frac{(1-\lambda)^{K-1}[(\bar{\lambda} - \lambda)(v - p + r) + \lambda r] + (1 - (1-\lambda)^{K-1})r}{1 - (1-\lambda)^{K-1}(\bar{\lambda} - \lambda)}, \tag{14}$$

$$g^* = \lambda(v - p + h(K)).$$

Proof. Without loss of generality, fix $h(0) = 0$. In order to proceed, we assume

$$v - p + h(K) \geq 0, \tag{15}$$

$$v - p + r + h(K) \geq h(i-1), \qquad \forall i = 2, \dots, K. \tag{16}$$

After we obtain the solution in the proposition, we will verify that inequalities (15) and (16) are both implied by (12). From the optimality equations (11), we have

$$g^* = \lambda(v - p + h(K)), \tag{17}$$

$$g^* + h(1) = \bar{\lambda}(v - p + r + h(K)), \tag{18}$$

$$g^* + h(i) = \lambda(v - p + r + h(K)) + (1-\lambda)h(i-1), \qquad \forall i = 2, \dots, K. \tag{19}$$

Taking the difference between (17) and (18), we obtain

$$h(1) = (\bar{\lambda} - \lambda)(v - p + r + h(K)) + \lambda r. \tag{20}$$

Taking the difference between (18) and (19) when $i = 2$, we have

$$\begin{aligned} h(2) &= (\lambda - \bar{\lambda})(v - p + r + h(K)) + (2 - \lambda)h(1) \\ &= (1 - \lambda)[(\bar{\lambda} - \lambda)(v - p + r + h(K)) + \lambda r] + \lambda r. \end{aligned} \tag{21}$$

Here, the second equality uses (20).

Taking the difference in equation (19) between i and $i - 1$ for $i \geq 3$ and rearranging, we obtain

$$h(i) = (2 - \lambda)h(i - 1) - (1 - \lambda)h(i - 2). \tag{22}$$

To show (13), first note that it holds for $i = 1, 2$. Next, we show by induction that it holds for $i = 3, \ldots, K$. Suppose it holds for $i - 1$ and $i - 2$. Using (22), we have

$$\begin{aligned} h(i) = {} & (2 - \lambda)\{(1 - \lambda)^{i-2}[(\bar{\lambda} - \lambda)(v - p + r + h(K)) + \lambda r] + (1 - (1 - \lambda)^{i-2})r\} \\ & - (1 - \lambda)\{(1 - \lambda)^{i-3}[(\bar{\lambda} - \lambda)(v - p + r + h(K)) + \lambda r] + (1 - (1 - \lambda)^{i-3})r\} \\ = {} & (1 - \lambda)^{i-1}[(\bar{\lambda} - \lambda)(v - p + r + h(K)) + \lambda r] + (1 - (1 - \lambda)^{i-1})r. \end{aligned}$$

Therefore, equation (13) holds for $i \geq 3$.

When $i = K$, equation (13) becomes

$$h(K) = (1 - \lambda)^{K-1}[(\bar{\lambda} - \lambda)(v - p + r + h(K)) + \lambda r] + (1 - (1 - \lambda)^{K-1})r.$$

Solving the above equation for $h(K)$ gives equation (14).

Next, we verify that the solution in the proposition satisfies inequalities (15) and (16). To verify (15), note that we have

$$v - p + h(K) = \frac{[1 - (1 - \lambda)^{K-1} + (1 - \lambda)^{K-1}\bar{\lambda}]r}{1 - (1 - \lambda)^{K-1}(\bar{\lambda} - \lambda)} \geq 0,$$

where the inequality follows from (12). To verify (16), we have for $i = 2, \ldots, K$,

$$
\begin{aligned}
&v - p + r + h(K) - h(i-1) \\
&= v - p + r + h(K) - (1-\lambda)^{i-2}[(\bar{\lambda} - \lambda)(v - p + r + h(K)) + \lambda r] - (1 - (1-\lambda)^{i-2})r \\
&= [1 - (1-\lambda)^{i-1}(\bar{\lambda} - \lambda)(v - p + r) + (1-\lambda)^{i-2})(1 - \bar{\lambda})r \\
&\geq 0.
\end{aligned}
$$

In the above, we used (15) to obtain the last inequality. This completes the proof. □

Note that Proposition 1 in our main analysis is a special case of Proposition 9 where $\bar{\lambda} = \lambda$. The condition (12) has very intuitive explanation. Reward expires if a customer purchases the product and earns a reward but does not come to the market for the subsequent K periods. In other words, the reward does not expire if the customer comes to the market within the expiration term, the probability of which is given by $1 - (1-\lambda)^{K-1} + (1-\lambda)^{K-1}\bar{\lambda}$. Note that the probability of coming to the market in the first $K-1$ periods is $1 - (1-\lambda)^{K-1}$ and the probability of not coming to the market in the first $K-1$ periods and coming to the market in the K-th period is $(1-\lambda)^{K-1}\bar{\lambda}$. Therefore, for given firm policy (p, r, K), the breakage is lower compared to the case without rush redemption. Customers take into account this lower breakage rate in their purchase decisions and condition (12) is less stringent than the one in the main analysis.

Corollary 1. *The value function $h(i)$ is an increasing function in i. Furthermore, g^* increases in $\bar{\lambda}$ and $h(i)$ increases in $\bar{\lambda}$ for each i.*

Proof. We show by induction that $h(i)$ is an increasing function in i. First, it is easy to verify that $h(1) \geq 0$. Rearranging equation (21), we have

$$h(2) = h(1) + (\lambda - \bar{\lambda})(v - p + r + h(K)) + (1-\lambda)h(1). \tag{23}$$

Since $h(2)$ equals to $h(1)$ plus a positive term, we have $h(2) \geq h(1)$. Now, rearranging equation (22), we obtain

$$h(i) = h(i-1) + (1-\lambda)(h(i-1) - h(i-2)). \tag{24}$$

Hence $h(i) \geq h(i-1)$ as long as $h(i-1) - h(i-2)$ is positive. Therefore, the result follows by induction.

To show that $h(i)$ increases in $\bar{\lambda}$, observe that $h(K)$ increases in $\bar{\lambda}$ since the numerator increases in $\bar{\lambda}$ and the denominator decreases in $\bar{\lambda}$. From (13), $h(i)$ increases in $\bar{\lambda}$ for each i. $\square$

Proposition 10. *Let q_i denote the stationary probability of state i for $i = 0, 1, \ldots, K$. We have*

$$q_i = \begin{cases} \frac{(1-\lambda)^K(1-\bar{\lambda})}{1-\lambda-(\bar{\lambda}-\lambda)[(1-\lambda)^K+\lambda]}, & \text{if } i = 0, \\ \frac{\lambda(1-\lambda)^{K-i+1}}{1-\lambda-(\bar{\lambda}-\lambda)[(1-\lambda)^K+\lambda]}, & \text{if } i = 1, \ldots, K, \end{cases} \qquad \forall i = 0, 1, \ldots, K.$$

Proof. The probability q_i's satisfy the following set of equations:

$$\begin{aligned} q_0 &= (1-\bar{\lambda})q_1 + (1-\lambda)q_0, \\ q_i &= (1-\lambda)q_{i+1}, \qquad\qquad \forall i = 1, \ldots, K-1, \\ q_K &= \lambda \sum_{i=2}^{K} q_i + \bar{\lambda}q_1 + \lambda q_0, \\ \sum_{i=0}^{K} q_i &= 1. \end{aligned}$$

Solving the equations above gives the result. $\square$

If we take $\bar{\lambda} = \lambda$, the stationary probabilities in Proposition 10 reduces to the form in Proposition 1 of the main analysis. Furthermore, we can show that q_i increases in $\bar{\lambda}$ for $i = 1, \ldots, K$ and decreases in $\bar{\lambda}$ for $i = 0$. Hence, a customer is less likely to be in the state with no reward as $\bar{\lambda}$ increases. This observation is not surprising; if customers engage in rush redemption when the reward is about to expire, they will earn more rewards.

The average revenue contribution of a customer is given by

$$\lambda \left[q_0 p + \sum_{i=2}^{K} q_i(p-r) \right] + \bar{\lambda}q_1(p-r) = \lambda \left[q_0 p + (1-q_0)(p-r) \right] + (\bar{\lambda}-\lambda)q_1(p-r). \tag{25}$$

Note that q_0 and q_1 depends on λ, $\bar{\lambda}$, and K, but not p and r. Therefore, the firm's decision problem stays the same except for slightly different coefficients. For given parameters, λ, $\bar{\lambda}$, and fixed expiration term K, the firm's optimization problem is again a linear programming problem. Therefore, the optimal solution is a corner solution, and our main analysis and results stays qualitatively the same. Unless $\bar{\lambda} = 1$, there will still be a region where one of the three market outcomes dominates. In particular, rush redemption does not invalidate our main analytical results and the effectiveness of reward programs. In fact, rush redemption does not necessarily reduce the profit of the seller; even though customers are more likely to purchase with rewards, they also purchase more often. This additional revenue contribution is shown in the last term of (25).

How does rush redemption affect the optimal expiration term? The answer depends on how rush redemption behavior differs between frequent and infrequent customers. Suppose that both segments share the same $\bar{\lambda}$; that is, the shopping probabilities of both frequent and infrequent customers jump to $\bar{\lambda}$ when the reward is about to expire (state 1), then rush redemption behavior has the effect of bringing closer the overall shopping probabilities of the two customer segments. In order to differentiate between frequent and infrequent customers, a longer expiration term is required. That said, all our results in the main analysis still hold qualitatively.

Online Appendix C: Consumer Discounting

Our main analysis assumes that consumers do not discount future surpluses. Here, we consider an extension where consumers discount future surpluses with a per-period discount factor $\delta \in [0, 1)$. We show that our main analysis applies to the model with consumer discounting under slight modification.

We again analyze the decision calculus of a generic consumer with the valuation v and shopping probability λ. The consumer's problem can be formulated as an infinite-horizon discounted-reward stochastic dynamic program. The objective is to maximize the total discounted consumer surplus. Recall the state $i \in \{0, 1, \ldots, K\}$. Let $u(\cdot)$ denote the value function. Then the optimality equation for the consumer's purchase decision can be written as

$$u(i) = \begin{cases} \lambda \max\{v - p + r + \delta u(K), \delta u(i-1)\} + (1-\lambda)\delta u(i-1), & \text{if } i = 1, \ldots, K, \\ \lambda \max\{v - p + \delta u(K), \delta u(0)\} + (1-\lambda)\delta u(0), & \text{if } i = 0. \end{cases}$$
$$\forall i = 0, 1, \ldots, K. \tag{26}$$

A consumer in state $i \geq 1$ purchases if and only if

$$v - p + r + \delta u(K) \geq \delta u(i-1) \quad \Leftrightarrow \quad \delta(u(K) - u(i-1)) \geq p - v - r.$$

Similarly, a consumer in state 0 purchases if and only if

$$v - p + \delta u(K) \geq \delta u(0) \quad \Leftrightarrow \quad \delta(u(K) - u(0)) \geq p - v.$$

As in our main model, a consumer will only stay in the market and contribute to the firm's long-run average revenue if she purchases in state 0. Otherwise, state 0 becomes an absorbing state and the consumer exits the market in the long run. Lemma 3 and Proposition 11 show that our main results in Section 3 carry over to the case with consumer discounting.

Lemma 3. *Suppose* $u(K) - u(0) \geq \delta(p - v)$, *then*

$$u(i) = u(0) + \lambda r \sum_{j=0}^{i-1} \delta^j (1-\lambda)^j = u(0) + \lambda r \Delta_i(\lambda), \qquad \forall i = 1, \ldots, K,$$

where

$$\Delta_i(\lambda) = \frac{1 - \delta^i (1-\lambda)^i}{1 - \delta(1-\lambda)}, \quad \forall i = 1, \ldots, K.$$

Proof. The proof is by induction. We have

$$\begin{aligned} u(1) &= \lambda \max\{v - p + r + \delta u(K), \delta u(0)\} + (1-\lambda)\delta u(0) \\ &= \lambda[v - p + r + \delta u(K)] + (1-\lambda)\delta u(0). \end{aligned} \tag{27}$$

In the above, the second equality uses $\delta(u(K) - u(0)) \geq p - v$. We also have

$$u(0) = \lambda[v - p + \delta u(K)] + (1-\lambda)\delta u(0). \tag{28}$$

Taking the difference between (27) and (28), we have

$$u(1) - u(0) = \lambda r.$$

Therefore, the lemma holds for $i = 1$.

For the inductive step, suppose the lemma holds for a given $i = 1, \ldots, K-1$. We have

$$\begin{aligned} u(i+1) &= \lambda \max\{v - p + r + \delta u(K), \delta u(i)\} + (1-\lambda)\delta u(i) \\ &= \lambda[v - p + r + \delta u(K)] + (1-\lambda)\delta u(i) \\ &= \lambda[v - p + r + \delta u(K)] + (1-\lambda)\delta u(i-1) + (1-\lambda)\delta[u(i) - u(i-1)] \\ &= u(i) + \lambda r \delta^i (1-\lambda)^i \\ &= u(0) + \lambda r \sum_{j=0}^{i} \delta^j (1-\lambda)^j. \end{aligned}$$

In the above, for the second equality, we used

$$\delta(u(K) - u(i)) = \delta(h(K) - h(0) - \lambda r\Delta_i) \geq p - v - r[1 - (1-\lambda)^i] \geq p - v - r.$$

This completes the proof. $\square$

Proposition 11. *Let*

$$g(i,\lambda) = \delta\lambda\Delta_i(\lambda), \qquad \forall i = 1,\ldots,K.$$

If $g(K,\lambda)r \geq p - v$, then the solution to the optimality equations (26) is given by

$$u(i) = \frac{\lambda}{1-\delta}[v - p + g(K,\lambda)r] + \lambda r\Delta_i(\lambda), \qquad \forall i = 0,1,\ldots,K. \tag{29}$$

In the above, we take $\Delta_0 = 0$. It is optimal for the consumer with the valuation v and shopping probability λ to purchase in each state.

What is the impact of consumer discounting on firm revenue, consumer welfare, and social welfare? Comparing the decision rules with and without discounting, a larger reward is required to induce consumer purchases. Hence, the firm's optimal revenue would be lower with consumer discounting. Also, it is less likely for the firm to offer a reward program.

Our result here should be contrasted with the case without discounting. Proposition 6 in the main text shows that without consumer discounting, offering a reward program never hurts frequent consumers regardless of their valuations, while the welfare impact of a reward program is mixed for infrequent consumers. Since it is less likely for the firm to offer a nontrivial reward under consumer discounting, consumer welfare for frequent consumers can only be lower with consumer discounting. The welfare on infrequent consumers can be higher or lower under consumer discounting.

Online Appendix D: Nontrivial Outside Option

Our main model assumes that the outside option has a value of 0. In this section, we consider an extension where the outside option has a non-trivial value $v_o \in [0, v_L]$. Conceptually, v_o can be perceived as the competition intensity: a larger (smaller) v_o means that the consumer's outside option is more (less) attractive. Optimality equations for the consumer's decision problem become

$$g^* + h(i) = \begin{cases} \lambda \max\{v - p + r + h(K), v_o + h(i-1)\} + (1-\lambda)h(i-1), & \text{if } i = 1, \ldots, K, \\ \lambda \max\{v - p + h(K), v_o + h(0)\} + (1-\lambda)h(0), & \text{if } i = 0, \end{cases}$$

$$\forall i = 0, 1, \ldots, K. \tag{30}$$

With some algebra, equation (30) can be rewritten as

$$g^* - \lambda v_o + h(i) = \begin{cases} \lambda \max\{v - v_o - p + r + h(K), h(i-1)\} + (1-\lambda)h(i-1), & \text{if } i = 1, \ldots, K, \\ \lambda \max\{v - v_o - p + h(K), h(0)\} + (1-\lambda)h(0), & \text{if } i = 0, \end{cases}$$

$$\forall i = 0, 1, \ldots, K. \tag{31}$$

Equation (31) is very similar to equation (1), with v replaced by $v - v_o$ and an additional constant term $-\lambda v_o$ on the left-hand side. Therefore, it can be similarly solved, and its solution is as follows. If $r\left[1 - (1-\lambda)^K\right] \geq p - v + v_o$, then an optimal solution to equation (31) is given by

$$h(i) = r\left[1 - (1-\lambda)^i\right], \qquad \forall i = 0, 1, \ldots, K, \tag{32}$$

$$g^* = \lambda\left(v + v_o - p + r\left[1 - (1-\lambda)^K\right]\right). \tag{33}$$

If $r\left[1 - (1-\lambda)^K\right] \leq p - v + v_o$, the consumer exits the market and does not contribute to the firm's revenue.

We briefly discuss the implications of a nontrivial outside option for consumers and the firm. Comparing (33) with (3), the average surplus earned by the consumer is λv_o higher per period if she stays in the market. This is quite intuitive, since the presence of a competing offer forces the firm to sweeten its offering in order to attract consumers. Therefore, the condition for retaining a consumer is more stringent than in the main model. Nevertheless, all results for the firm's decision problem hold with v_H and v_L replaced by $v_H - v_o$ and $v_L - v_o$, respectively. From Lemma 1, the firm's profit is strictly increasing in consumer valuations. With v_H and v_L replaced by $v_H - v_o$ and $v_L - v_o$, the firm's profit decreases in each market outcome compared with the main model. In the market outcome with a reward (part (iii) in Lemma 1), the optimal reward depends either on the valuation differential or v_L. With v_H and v_L replaced by $v_H - v_o$ and $v_L - v_o$, the optimal reward either stays the same or decreases. In other words, for a fixed expiration term K, the optimal reward is the same or lower in the presence of a nontrivial outside option. To understand the rationale for offering a lower reward, we note that the optimal price is also lower in the presence of a nontrivial outside option. The optimal expiration term K^* in Proposition 3 is an increasing function of the ratio v_H/v_L. With v_H and v_L replaced by $v_H - v_o$ and $v_L - v_o$, the optimal expiration term increases as $(v_H - v_o)/(v_L - v_o) \geq v_H/v_L$. Intuitively, a nontrivial outside option forces the firm to offer longer expiration terms. Furthermore, we notice that when v_H/v_L is small to start with, greater competition (v_o) leads to $(v_H - v_o)/(v_L - v_o)$ being larger than v_H/v_L, which gives the seller a stronger incentive to offer a reward program. Thus, competition constitutes a possible explanation of the empirical finding that sellers with low price margin are also likely to offer reward programs.

To conclude, in the presence of a nontrivial outside option, i) consumers are better off, ii) for a fixed expiration term, a lower reward is offered together with a lower price, and iii) the optimal expiration term is longer. It would be interesting to investigate whether the same conclusions hold in an explicit, dynamic game-theoretical model. We leave such an extension to future research.

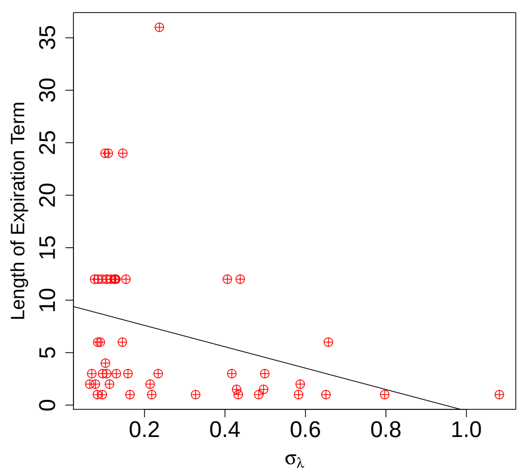
Online Appendix E: Empirical Analysis

Table 3: Summary Statistics

	Nobs	Mean	Std.dev.	Min	Max
1. Retailer characteristics					
US sales in 2013 ($billions)	100	18.58	36.82	3.60	334.30
Number of stores (in 1,000)	100	2.71	4.04	0	26.64
Number of employees (in 1,000)	100	111.90	239.63	0.53	2,200
2. Reward program characteristics					
Has rewards program	98	0.56	0.49	0	1
Reward size (%)	55	6.58	6.93	1	33
Reward expires	55	0.84	0.37	0	1
Expiration term (months)	47	6.98	7.70	1	36
3. Purchase frequencies					
Mean purchase frequency (λ_{mean})	97	0.181	0.131	0.048	0.555
Sd purchase frequency (σ_λ)	97	0.284	0.235	0.053	1.082
Log ratio of purchase frequency (Φ_λ)	97	0.439	0.595	0.047	3.160

Figure 5: Expiration Term vs. Purchase Frequency Heterogeneity

(a) Reward Expiration Term vs. σ_λ



(b) Reward Expiration Term vs. Φ_λ