Managing Clearance Sales in the Presence of Strategic Customers

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We study the effect of strategic customer behavior on pricing and rationing decisions of a firm selling a single product over two periods. The seller may limit the availability of the product (that is, ration) in the second (clearance) period. Some customers are strategic and respond to the firm's decisions by timing their purchases. When capacity is nonconstraining and the seller has pricing flexibility, we show that rationing in the clearance period cannot improve revenue. However, when prices are fixed in advance, rationing can improve revenue. In the latter case, we conduct a detailed analysis for linear and exponential demand curves and derive explicit expressions for optimal rationing levels. We find that the policy of doing the better of not restricting availability at the clearance price or not offering the product at the clearance price is typically near optimal. Our analysis also suggests that rationing—although sometimes offering considerable benefit over allowing unrestricted availability in the clearance period—may allow the seller to obtain only a small fraction of the optimal revenue when the prices are chosen optimally without rationing. We extend the analysis to cases where the capacity is constraining and obtain similar results.

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1. Introduction

In situations where a firm dynamically manages the price and availability of a product, customers may try to time their purchases to maximize their own benefit. Although there is a considerable body of research on dynamic pricing and revenue management in the operations management literature, the majority of this work does not explicitly model such strategic customer behavior. In this paper we analyze a model for selling stock of a single product in two periods—a first "regular" sales period, followed by a second "clearance" period. In the model, a fraction of customers is strategic and will buy either in the first period or in the second period to maximize their own expected surplus. The firm uses price or availability or both to induce a desirable customer response to maximize its own revenue.

Recently, the Internet has been used by airlines to sell "distressed inventory"—seats that remain unsold close to a flight's departure date—at a discount. Examples include the Last Minute Packages on the popular travel site Travelocity.com, the airline-backed travel site Hotwire.com, and various last-minute offers from airline websites, such as CyberSaver fares offered by Northwest Airlines. Similar last-minute sales are also prevalent in the hotel industry. An example is the Weekend Getaways program offered on starwoodhotels.com. By selling such distressed inventory, additional revenue is garnered on seats or hotel rooms that would otherwise go empty. However, a new problem arises; the availability of last-minute discounts may dilute full-price demand in the regular selling period. Clearance sales are also prevalent in retail practice, where significant discounts are often given for items that are left over at the very end of a selling season. For example, many retailers cut prices sharply on December 26th, immediately after the Christmas shopping season. In response to retailers' selling strategies, customers may respond strategically and choose to wait for the sale. Consequently, revenue at regular prices may decrease.

Taking the seller's perspective, we develop a model that captures the trade-off between getting more revenue from last-minute low-price sales and losing revenue from regular sales when some customers wait to buy at the low price. In view of customers' strategic behavior, the firm is potentially interested in rationing inventory (capacity) at the low price; that is, it may

opt to satisfy only a fraction (between 0 and 1, inclusive) of demand at the low price to deter customers otherwise willing to pay the high price from waiting for the discount.

We study how the ability to change prices affects decisions and revenue. In particular, we show that if the seller has ample capacity (throughout, we use the term *capacity* to refer to the total amount of inventory available for potential sale) and can optimize prices, then there is no additional benefit to restricting availability in the clearance period, provided prices are chosen optimally. We provide detailed analysis of situations where total demand is linear or exponential in price and show there that if the firm ignores strategic customer behavior, then it sets prices too low. The revenue loss from such pricing policies can be as high as 11% of the optimal revenue. However, when only a small proportion (less than 30%) of customers strategically times purchases, the revenue loss to the seller is quite small (less than 1%).

When prices are fixed, the seller generally does benefit from rationing. We describe situations in which the rationing decision simplifies considerably, and it is best either to not sell in the clearance period or to sell in the clearance period without placing any restriction on availability. In the former case, the rationing level is 0, and in the latter it is 1. We again focus on linear and exponential demand curves. For each, we derive optimal rationing decisions and completely characterize those cases where the rationing level is nontrivial (that is, the optimal fraction of demand to satisfy at the clearance price is strictly between 0 and 1). Our computational results show that even when the optimal rationing level is nontrivial, setting the rationing level either to the better of 0 (effectively selling only in the regular sales period) or 1 (selling in the regular period and also in the clearance period without restriction) yields revenue that is close to that which would be obtained using an optimal rationing level. This suggests it is fairly safe to consider price-only policies.

We also study the effectiveness of rationing at compensating for a poor choice of prices. If the fixed prices differ from those that would have been chosen in the case of pricing flexibility, then as mentioned above, rationing will, in general, increase revenue. However, not surprisingly, the revenue obtained even when the rationing level is set optimally for the fixed prices can be far from that which would have been obtained if the prices were instead fixed at an optimal level. Nevertheless, in many instances, rationing—as opposed to allowing unrestricted availability to the leftover inventory—offers considerable benefits.

We consider settings where capacity is constraining. When there is pricing flexibility, we derive an upper bound on the seller's revenue. For linear and exponential demand curves, we show that the bound is tight and that as in the ample capacity setting, there is no added benefit to rationing in the clearance period. However, in contrast to the ample capacity setting, the latter conclusion does not hold for all demand curves.

We also address the case where capacity is constraining and the prices are fixed. Our numerical investigation suggests that a simple "two-extreme policy" that does the better of selling as much as possible at the low price or not selling at all at the low price is close to optimal. (With finite capacity, it may not be possible to satisfy all demand in the clearance period.) For both linear and exponential demand curves, the revenue loss from such a policy compared with an optimal rationing policy is small (less than 3%). If strategic customer behavior is ignored, the seller will not restrict sales of leftover inventory in the clearance period. Our numerical study shows that such a policy can result in a loss of up to 70% of the optimal revenue. This emphasizes the importance of considering both extremes in the two-extreme policy and also shows the importance of rationing as a response to strategic customer behavior.

The remainder of the paper is organized as follows. Section 2 reviews related literature. Section 3 studies the case with infinite capacity and pricing flexibility. Section 4 studies the case with infinite capacity and fixed prices. Section 5 studies the case with finite capacity and pricing flexibility. Section 6 studies the case with finite capacity and fixed prices. Section 7 provides a summary. All proofs are given in the appendix.

2. Related Literature

Clearance pricing has been studied extensively in the operations literature. Here we do not attempt a complete review. Gupta et al. (2006) consider a clearance pricing model where there are multiple markdown opportunities, and Pundoor et al. (2005) study the markdown pricing problem faced by a firm that owns multiple stores. Their papers point to other recent work in the area. Bitran and Caldentey (2003) and Elmaghraby and Keskinocak (2003) also provide surveys. Petruzzi and Dada (1999) review newsvendor problems with pricing. Most existing clearance pricing models do not consider strategic customer behavior.

There is a vast literature on revenue management that addresses questions about managing prices and availability of airline seats; see Talluri and van Ryzin (2004). Shen and Su (2007) survey the emerging area of customer behavior modeling for revenue management. Until recently, most revenue management models have not considered strategic customer behavior

and have not taken into account potential long-term effects that revenue management decisions may have on future demand patterns. In most revenue management research, the focus is on analyzing detailed models of the regular booking process. The approach in our paper is instead to consider a setup that collapses the "regular" booking horizon into a single time period. This allows us to explicitly incorporate strategic customers while maintaining a tractable model suitable for generating insights.

The work most closely related to ours is that of Gallego et al. (2004) and Liu and van Ryzin (2005). Our model of strategic customer behavior is in essence the same as that of Gallego et al. in their fluid model and of Liu and van Ryzin in their risk-neutral case. Gallego et al. consider two-period models in which the decision is the sell-up-to limit in the second period. They do not consider pricing problems; however, they do explicitly model stochastic demand and dynamic updating of consumer expectations. Liu and van Ryzin also consider a situation in which a single product is sold at two different prices in two periods. In contrast to our work, however, they consider the decision of how much inventory to order prior to the regular selling season, but the rationing level itself is not a decision variable. Instead, they consider policies that do not place a limit on availability of the product in the clearance period beyond that imposed by the initial choice of inventory. This should be contrasted to our work in which initial inventory is fixed and the decision variable is the rationing level. Our analysis of fixed capacity problems is most applicable to cases where either (a) inventory ordering decisions are decoupled from pricing and availability decisions because they are made by different organizational groups, or (b) initial inventory cannot easily be changed, as in the case of an airline flight once the aircraft is assigned. Yet another distinction is that we suppose a portion of the customers is strategic and a portion myopic, which can lead to qualitatively different answers than considering only strategic customers as in Liu and van Ryzin and in Gallego et al. Liu and van Ryzin also consider risk-averse customers and competition, both of which we do not address.

Su (2005) considers dynamic intertemporal pricing strategies for a seller facing customers with two different possible valuations. A fraction of the customers tries to strategically time purchases. The problem is formulated as a two-stage game, and demand is modeled as a deterministic continuous fluid that flows in over time. The central results are expressions for the seller's optimal pricing and rationing paths. The expressions yield conditions under which a decreasing (or increasing) price path is optimal. Ovchinnikov and Milner (2005) study a problem in which a firm sells identical products at two or three predetermined prices over a series of selling periods. They develop a model that captures trade-offs between short-term gains from last-minute sales and longer-term losses from customers learning to expect such sales. They model the consumer learning process and consider stochastic demand. Under some assumptions, they present results on the structure of the optimal sales policy. Popescu and Wu (2007) address a problem in which customers dynamically update their response to a firm's pricing policy over a series of selling seasons.

Aviv and Pazgal (2008) study the dynamic pricing problem of a seller with finite capacity that faces forward-looking customers who arrive over a finite horizon. Given a price path, customers form expectations on the product availability in the future and time their purchases accordingly. They study the impact of customer behavior on the pricing policies of the seller. Caldentey and Vulcano (2007) study a revenue management problem in which customers either buy at a list price or participate in an auction, the result of which is revealed at the end of the selling horizon. They analyze the equilibrium purchase strategy for customers. In their paper, auctioning serves as a capacity allocation mechanism. Both of these papers consider trade-offs regarding the timing of a purchase from customers' perspectives and both provide models that take into account the effects of strategic customer behavior for the firm's revenue management decisions. Elmaghraby et al. (2008) consider a markdown pricing mechanism where prices are set according to a preannounced schedule. In their setup, there are multiple units of a single product with a known number of customers, each of which demands multiple units.

Our paper is also related to work that considers intertemporal price discrimination; see Stokey (1979) for some early research in this area. Lazear (1986) considers intertemporal pricing behavior under various assumptions on product characteristics and market conditions. Besanko and Winston (1990) consider the intertemporal pricing problem for a monopolist selling a new product in several periods under the assumption that customers are intertemporal utility maximizers. They model the problem as a game and characterize a subgame perfect Nash equilibrium pricing policy. In their problem, customer buying behavior is driven by utility discounting over time (high utility customers lose more if they wait; therefore, they are willing to buy early and pay more); product availability is not an issue because it is assumed that the product can be produced at constant marginal cost and production capacity is infinite. Nocke and Peitz (2005) argue from a mechanism design perspective that clearance pricing can be the best intertemporal selling strategy in certain situations.

Rationing has been the topic of much research in the economics literature. Wilson (1989) studies priority service contracts as an alternative to spot prices and fixed prices. His work shows that substantial efficiency gains can be obtained using priority service in the absence of spot markets. Gilbert and Klemperer (2000) study a rationing model in which customers must make a sunk investment to enter the market. They show that even in a single-period model, prices that result in rationing may be more profitable than market-clearing prices. Both papers review the economics literature on rationing.

Our models are static in the sense that they consider the system under steady state operating conditions and do not consider a dynamic process that leads to such conditions. Nevertheless, modeling dynamic customer learning may help in understanding interactions of customer strategy and the seller's decision process. Phillips (2000) incorporates customer-switching behavior into a model for the management of distressed inventory in the airline industry. (His model for how customers decide between purchasing at the regular or clearance price is nearly identical to ours.) Zohar et al. (2002) consider a queueing system in which customers adjust their abandonments time according to their expectations. They analyze the system in steady state and propose a dynamic learning model where customer expectations are formed through accumulated experience. They show through simulation that the system converges to the anticipated equilibrium.

3. Infinite Capacity with Pricing Flexibility

Consider a seller of a single product. In this section, we suppose capacity is infinite and the seller has pricing flexibility. Demand for the product is deterministic and depends upon the price of the product. This dependency is summarized by the demand curve $\{D(p): p \geq 0\}$ where $D(p) < \infty$ for all p; if the price of the product is p, then demand for the product is $D(p)$. We assume $D(p)$ is nonincreasing and $\lim_{p\to\infty} pD(p) = 0$. The demand curve is assumed to arise as follows. There is a population of customers, each of whom has an individual *maximum willingness to pay* (MWP) for the product. The demand $D(p)$ is the number of customers in the population with MWP greater than p. Throughout, we assume that demand is a continuous quantity.

Suppose the product is first offered for sale at price p_1 and then later offered for sale at price p_2 (with $p_2 \leq p_1$) in a second "clearance period." There are two types of customers: myopic and strategic. A *myopic* customer buys the product in the first period if and only if the price p_1 is below his MWP; myopic cus-

tomers with MWPs lower than p_1 but higher than p_2 purchase in the second period. A *strategic* customer times his purchase to maximize his consumer surplus, which is the difference between his MWP and the purchase price. Strategic customers are also assumed to know what the seller's clearance price p_2 will be, even before the clearance period. Although admittedly a strong assumption, it is somewhat close to reality in some situations. For example, frequent shoppers may have a fairly good idea of the price path for the items they are interested in. Suppose that among all customers a proportion α is myopic. The remaining proportion $\bar{\alpha} = 1 - \alpha$ is strategic. We assume that whether or not a customer is myopic or strategic is independent of his MWP. At this stage, we are not considering the possibility that the seller will limit availability in the clearance period (this will be considered later), so all strategic customers will wait to buy at the price p_2 because $p_2 \leq p_1$.

In this setting, the sales at price p_1 will be the number of myopic customers with valuations above p_1: $\alpha D(p_1)$. The sales at price p_2 will be the number of customers with valuations below p_1 but above p_2, plus the number of strategic customers with valuations above p_1: $[D(p_2) - D(p_1)] + \bar{\alpha}D(p_1) = D(p_2) - \alpha D(p_1)$. The seller's revenue is

$$R(p_1, p_2) = \alpha p_1 D(p_1) + p_2[D(p_2) - \alpha D(p_1)]. \quad (1)$$

The maximum revenue obtainable for the firm is

$$v^\dagger = \max_{p_1 \geq p_2 \geq 0} R(p_1, p_2). \quad (2)$$

Note that we allow $p_1 = p_2$, in which case a single price is effectively used. It can be seen that the revenue $v^\dagger$ is increasing in the proportion of myopic customers α.

Suppose now that it is possible for the seller to control both the prices and the availability of the product. In particular, it can limit availability at the lower price, so that each customer seeking to buy at price p_2 can obtain the product with probability $\theta \in [0, 1]$. Note that we do not make a distinction between (a) satisfying a proportion θ of demand at price p_2 and (b) offering the product at the lower price p_2 with probability θ. In our deterministic context, such a distinction is not important because the two alternatives lead to the same customer response. We will term the act of selecting θ as *rationing*.

We assume strategic customers are risk neutral and seek to maximize their expected surplus. Hence, a strategic customer with MWP u buys at price p_1 if and only if

$$u - p_1 \geq \theta(u - p_2)^+. \quad (3)$$

If $\theta \in [0, 1)$, condition (3) is equivalent to $u \geq (p_1 - \theta p_2)/(1 - \theta)$. If $\theta = 1$, (3) reduces to

$$u - p_1 \geq u - p_2, \quad u \geq p_1. \tag{4}$$

Condition (4) implies that either u is infinite or $\infty > u \geq p_1 = p_2$. Let

$$r(\theta, p_1, p_2) = \begin{cases} \dfrac{p_1 - \theta p_2}{1 - \theta} & \text{if } 0 \leq \theta < 1 \\ p_1 & \text{if } p_1 = p_2,\ \theta = 1 \\ \infty & \text{if } p_1 > p_2,\ \theta = 1. \end{cases} \tag{5}$$

Summarizing the above, a strategic customer buys at price p_1 if and only if $u \geq r(\theta, p_1, p_2)$. Similarly, a strategic customer buys at price p_2 if and only if $p_2 \leq u < r(\theta, p_1, p_2)$. Note that $r(\theta, p_1, p_2)$ as defined in (5) is continuous in θ. The above model of strategic customer behavior is essentially that proposed by Phillips (2000) and later used by Gallego et al. (2004) and Liu and van Ryzin (2005).

Given θ, p_1, and p_2, the number of customers that buys at price p_1 is

$$d_1(\theta, p_1, p_2) = \alpha D(p_1) + \bar{\alpha} D(r(\theta, p_1, p_2)), \tag{6}$$

and the number of customers that wants to buy at p_2 is

$$d_2(\theta, p_1, p_2) = D(p_2) - \alpha D(p_1) - \bar{\alpha} D(r(\theta, p_1, p_2)). \tag{7}$$

Note that in settings where customers are risk averse, or where they incur a cost for waiting to make a purchase, the demand in the clearance (regular) period will be less (greater) than that specified by the model above. Analysis of this issue is beyond the scope of this paper. The revenue from selling at prices p_1 and p_2 and rationing level θ is

$$\Psi(\theta, p_1, p_2) = p_1 d_1(\theta, p_1, p_2) + \theta p_2 d_2(\theta, p_1, p_2) \tag{8}$$

$$= R(p_1, p_2) + (1 - \theta)\Omega(\theta, p_1, p_2), \tag{9}$$

where

$$\Omega(\theta, p_1, p_2) = \widehat{\Omega}(p_1, p_2, r(\theta, p_1, p_2)) \tag{10}$$

and

$$\widehat{\Omega}(p_1, p_2, q) = \bar{\alpha} q D(q) - p_2[D(p_2) - \alpha D(p_1)]. \tag{11}$$

Note that $\Psi(1, p_1, p_2) = R(p_1, p_2)$ and $\Psi(0, p_1, p_2) = p_1 D(p_1)$. The optimal revenue is

$$V = \max_{\theta \in [0, 1],\, 0 \leq p_2 \leq p_1} \Psi(\theta, p_1, p_2).$$

Our first objective is to compare V with $v^\dagger$.

PROPOSITION 1. *Suppose that capacity is infinite. If the seller can choose the prices, then restricting availability cannot improve the total revenue; that is, $V = v^\dagger$.*

The proposition shows that if the firm can control prices, it gets no benefit from rationing. Put differently, it is optimal to set $\theta = 1$, provided prices are chosen appropriately. Note also that because $v^\dagger$ is increasing in α, so is V; i.e., when more customers are strategic, the seller obtains less revenue.

3.1. Linear Demand Curve
The linear demand curve is given by

$$D(p) = \begin{cases} 1 - p & \text{if } 0 \leq p \leq 1 \\ 0 & \text{if } p > 1. \end{cases} \tag{12}$$

Our analysis can be adapted to cases in which $D(p) = a - bp$ if $0 \leq p \leq a/b$ and $D(p) = 0$ otherwise for $a, b > 0$. The linear demand curve corresponds to a fixed customer population with uniformly distributed MWPs as in Gallego et al. (2004) and Liu and van Ryzin (2005).

Suppose the seller can set the prices. Then the optimization problem to be solved is

$$v^\dagger = \max_{0 \leq p_2 \leq p_1 \leq 1} \alpha p_1(1 - p_1) + p_2[1 - p_2 - \alpha(1 - p_1)].$$

From the first-order conditions, an optimal solution is

$$(p_1^\dagger, p_2^\dagger) = \left(\frac{3 - \alpha}{4 - \alpha}, \frac{2 - \alpha}{4 - \alpha}\right), \qquad v^\dagger = \frac{1}{4 - \alpha}.$$

If all customers behave strategically ($\alpha = 0$), then selling at two prices will give the same revenue as selling at a single price because all customers will buy at the lower price. Note that both $p_1^\dagger$ and $p_2^\dagger$ are increasing in the proportion of myopic customers α. Later we will see that similar observation holds for exponential demand curve.

What happens if the seller incorrectly assumes that all customers are myopic (i.e., $\alpha = 1$) and tries to optimize prices? Then the prices chosen will be $\tilde{p}_1 = 2/3$ and $\tilde{p}_2 = 1/3$, giving revenue $\tilde{v}^\dagger = R(2/3, 1/3) = (\alpha + 2)/9$, which is less than the optimal $v^\dagger = 1/(4 - \alpha)$. Observe that $\tilde{p}_i < p_i^\dagger$ for $i = 1, 2$ when $\alpha \in [0, 1)$; that is, ignoring strategic behavior results in prices that are too low. The relative performance gap between the revenue accrued from the optimal pricing policy and that from the policy that ignores strategic customer behavior is given by $(v^\dagger - \tilde{v}^\dagger)/v^\dagger = (1 - \alpha)^2/9$. If α is large, the gap is small. In particular, if $\alpha \geq 0.7$, the gap is less than 1%. However, if α is small, the gap can be as large as about 11% (at $\alpha = 0$). This indicates that to appropriately set prices, it is important to account for the presence of strategic customers. But when there is only a small fraction of strategic customers, it is fairly safe to ignore strategic customer behavior.

3.2. Exponential Demand Curve
The exponential demand curve is given by

$$D(p) = e^{-p}. \tag{13}$$

The analysis can be extended to $D(p) = ae^{-bp}$ for a, $b > 0$. The firm faces the problem

$$\max_{p_1 \geq p_2 \geq 0} \alpha p_1 e^{-p_1} + p_2(e^{-p_2} - \alpha e^{-p_1}),$$

for which an optimal solution is

$$(p_1^\dagger, p_2^\dagger) = \left(2 - \frac{\alpha}{e}, 1 - \frac{\alpha}{e}\right), \qquad v^\dagger = e^{\alpha/e - 1}.$$

If the firm ignores strategic customer behavior and assumes $\alpha = 1$, then it will choose prices $\tilde{p}_1 = 2 - 1/e$ and $\tilde{p}_2 = 1 - 1/e$ and the revenue will be $\tilde{v}^\dagger = (\alpha + e - 1)e^{1/e - 2}$. As in the linear case, we have $\tilde{p}_i < p_i^\dagger$ for $i = 1, 2$ when $\alpha \in [0, 1)$. The relative revenue gap $(v^\dagger - \tilde{v}^\dagger)/v^\dagger$ here behaves similarly to that for the linear demand curve.

4. Infinite Capacity with Fixed Prices
In this section, we again assume the capacity of the seller is infinite, but the product is sold at fixed prices p_1, p_2 with $p_1 > p_2$. Because prices are fixed, we suppress (p_1, p_2) from notation; e.g., $R = R(p_1, p_2)$, $\Psi(\theta) = \Psi(\theta, p_1, p_2)$, etc. This convention is followed throughout the paper. Without rationing, the total revenue is R given in (1). With rationing, the seller can control the availability of the product at price p_2 by choosing θ. The maximum revenue is

$$v^{\mathrm{FP}} = \max_{\theta \in [0, 1]} \Psi(\theta). \tag{14}$$

The following propositions examine when rationing does and does not yield benefits.

PROPOSITION 2. *Suppose the product is sold at two fixed prices $p_1 > p_2$. If there exists price q with $q \geq p_1$ such that*

$$\widehat{\Omega}(p_1, p_2, q) > 0, \tag{15}$$

then rationing strictly improves revenue over not rationing; i.e., $v^{\mathrm{FP}} > R = \Psi(1)$. If $\widehat{\Omega}(p_1, p_2, q) \leq 0$ for all $q \geq p_1$, then rationing does not improve revenue; i.e., $v^{\mathrm{FP}} = R$.

PROPOSITION 3. *Suppose the product is sold at two fixed prices $p_1 > p_2$. If p_1 satisfies*

$$p_1 D(p_1) \geq p D(p) \quad \text{for all } p \geq p_1, \tag{16}$$

then rationing does not improve revenue over the better of selling at the prices (p_1, p_2) without rationing and selling at only price p_1; that is,

$$v^{\mathrm{FP}} = \max\{R, p_1 D(p_1)\} = \max\{\Psi(1), \Psi(0)\}.$$

Put differently, the optimal rationing level is either 0 or 1.

In §§4.1 and 4.2, we specialize the analysis to linear and exponential demand curves.

4.1. Linear Demand Curve
We assume prices are fixed with $0 < p_2 < p_1 < 1$. Define $z = [\alpha p_2(p_1 - p_2)/\bar{\alpha}]^{1/2}$ and

$$\hat{\theta} = \frac{1 - p_1}{1 - p_2}. \tag{17}$$

The quantity $\hat{\theta}$ will play a key role in our analysis of both linear and exponential demand curves. Note that $r(\hat{\theta}) = 1$.

PROPOSITION 4. *Suppose $D(p)$ is linear as in (12) and $0 < p_2 < p_1 < 1$.*
(i) If $\alpha = 1$, then $v^{\mathrm{FP}} = R$, and an optimal rationing level is $\theta^* = 1$.
(ii) If $\alpha = 0$, then

$$v^{\mathrm{FP}} = \begin{cases} p_2(1 - p_2) & \text{if } p_1 + p_2 \geq 1 \\ p_1(1 - p_1) & \text{if } p_1 + p_2 < 1, \end{cases}$$

and an optimal rationing level is

$$\theta^* = \begin{cases} 1 & \text{if } p_1 + p_2 \geq 1 \\ 0 & \text{if } p_1 + p_2 < 1. \end{cases}$$

(iii) If $\alpha \in (0, 1)$, then

$$v^{\mathrm{FP}} = \begin{cases} p_1(1 - p_1) \\ \qquad \text{if } z < p_1 - p_2, \text{ and } p_1(1 - p_1) \geq R \\ R + 2\bar{\alpha}(p_1 - p_2)\left[\frac{1}{2} - p_2 - z\right] \\ \qquad \text{if } p_1 - p_2 \leq z \leq \frac{1}{2} - p_2 \\ R \quad \text{otherwise} \end{cases} \tag{18}$$
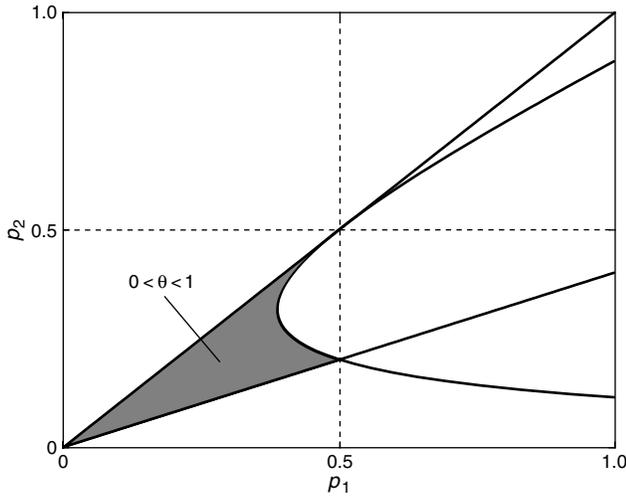
and an optimal rationing level is

$$\theta^* = \begin{cases} 0 & \text{if } z < p_1 - p_2, \text{ and } p_1(1 - p_1) \geq R \\ 1 - \sqrt{\dfrac{\bar{\alpha}(p_1 - p_2)}{\alpha p_2}} & \text{if } p_1 - p_2 \leq z \leq \frac{1}{2} - p_2 \\ 1 & \text{otherwise.} \end{cases} \tag{19}$$

Observe that part (ii) implies that the product is sold only at one price when all customers are strategic, because if $\theta^* = 1$, then no customer will buy at price p_1, and if $\theta^* = 0$, then the product is only offered at price p_1. The same conclusion emerges as a consequence of Proposition 3 of Gallego et al. (2004) and of Proposition 3 of Liu and van Ryzin (2005).

Continuing the discussion of our model with $\alpha = 0$, Proposition 4 provides sufficient conditions for the "triviality" of the rationing level. In §4.2 we will see

Figure 1   **Region Where $0 < \theta < 1$ for Price Pair (p_1, p_2) and**
  $\alpha \in [1/4, 1)$

Figure 2   **Revenue $\Psi(\theta)$ as a Function of θ for Linear Demand Curve;**
 $p_1 = 0.3$, $p_2 = 0.1$, $\alpha = 0.8$

that a similar result holds for a exponential demand curve; that is, either 0 or 1 is an optimal rationing level when $\alpha = 0$. Hence, one might ask if this is the case in general whenever $\alpha = 0$. The answer is no. For instance, if $D(p) = 1.8 - 3p$ for $0 \le p \le 0.4$, $D(p) = 1 - p$ for $0.4 < p \le 1$, and $D(p) = 0$ otherwise, then it can be checked for fixed prices $(0.4, 0.15)$ that an optimal rationing level is 0.09, giving revenue of 0.2406, and $\Psi(0) = 0.24$, $\Psi(1) = 0.2025$. In a similar vein, Gallego et al. (2004) give examples where "mixed strategies," in which all leftover inventory is offered for sale with a probability strictly between 0 and 1, are optimal.

For $\alpha \in (0, 1)$, observe that by (19), if $p_1 \ge 1/2$, then θ^* is either 0 or 1. This observation coincides with the result in Proposition 3. For a linear demand curve, we have $p_1 D(p_1) \ge pD(p)$ for all $p \ge p_1$ if $p_1 \ge 1/2$. Therefore, the condition (16) is satisfied if $p_1 \ge 1/2$. Figure 1 shows the values of p_1 and p_2 for which the rationing decision is nontrivial; i.e., when $\theta^* \notin \{0, 1\}$. The lines are determined by the conditions in Proposition 4 Part (iii) when $\alpha \in [1/4, 1)$. The picture is similar when $\alpha \in (0, 1/4)$. From the figure, we can see that the set of price pairs that give a nontrivial rationing level is fairly small. In general, the rationing decision is nontrivial when the high and low prices are close and small.

Figure 2 shows revenue $\Psi(\theta)$ as a function of θ for a case where the rationing decision is nontrivial ($p_1 = 0.3$, $p_2 = 0.1$, $\alpha = 0.8$). The function is concave on $[0, 7/9]$ and linear on $[7/9, 1]$. (We found a similar structure for other types of demand curves as well.) When θ is greater than 7/9, no strategic customer will buy at price p_1; hence, the total revenue is increasing linearly in θ in this range. This occurs because increases in θ within $[7/9, 1]$ allow the firm to sell more items to myopic customers at price p_2 while not
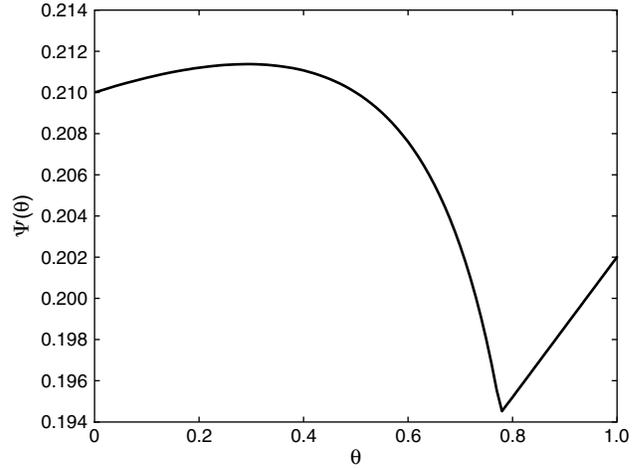
suffering any additional cannibalization at price p_1. In the figure, the optimal value of θ is $1 - \sqrt{2}/2 \approx 0.2929$, yielding revenue of approximately 0.2114. If we set $\theta = 0$, then the corresponding revenue is $p_1 D(p_1) = 0.3 \times 0.7 = 0.21$, which is only slightly below optimal.

4.1.1.   The Effect of Rationing. To study the effectiveness of rationing at overcoming suboptimal pricing decisions, we compare v^{FP} with $v^\dagger$. (Recall that $v^\dagger$ is the revenue accrued from optimal prices and that for such optimal prices, rationing provides no additional benefit to the firm.) Suboptimal prices might be used if, for instance, prices are set by one group within the firm and availability decisions are made by another. This is sometimes the case in airline revenue management. Figure 3 shows the ratio $v^{\text{FP}}/v^\dagger$ as a function of (p_1, p_2) when $\alpha = 0.5$. It can be seen that when prices take extreme values (at the "corners" in the figure), then $v^{\text{FP}}/v^\dagger$ can be arbitrarily close to 0; i.e., v^{FP} is far from $v^\dagger$. Not surprisingly, $v^{\text{FP}}/v^\dagger$ is very

Figure 3 $v^{\text{FP}}/v^\dagger$ **as a Function of (p_1, p_2) When $\alpha = 0.5$**
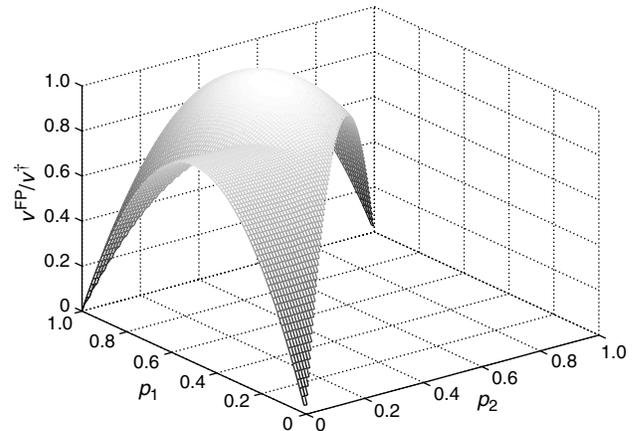
Figure 4   Relative Performance of Different Policies as a Function of p_2 when $\alpha = 0.5$ and $p_1 = 0.4$
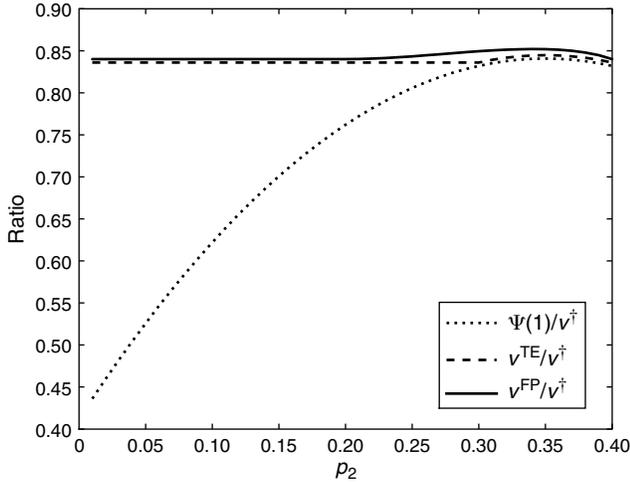
small if both prices are set too high (p_1 and p_2 both near 1 in the figure) or too low (p_1 and p_2 both near 0 in the figure). Likewise, if p_1 is too high and p_2 is too low, there is little rationing can do to make up for the poor pricing because restricting sales at p_2 will cause most customers to elect not to buy rather than to buy at p_1. Note also that the plot lies strictly below 1 except at $(p_1, p_2) = (5/7, 3/7)$, which are the prices that maximize $R(p_1, p_2)$. Hence, rationing never completely overcomes suboptimal pricing.

Figure 4 shows the relative performance of several policies when $\alpha = 0.5$ and $p_1 = 0.4$. The value $\Psi(1)$ is the revenue when sales in the clearance period are not restricted (that is, when the rationing level is $\theta = 1$—the value that the firm would choose if it acted as if all customers were myopic). The value $v^{\text{TE}} = \max\{\Psi(0), \Psi(1)\}$ is the revenue obtained by doing the better of the two extremes, setting the rationing level to 0 or to 1. The revenue v^{FP} is that from using optimal rationing levels. There are several observations here. First, note that even when an optimal rationing policy is used for the given prices, the revenue obtained is at most about 85% of $v^\dagger$. Second, the rationing level $\theta = 1$ here performs very badly for low values of p_2, whereas $\theta = 0$ performs well for such values. Not surprisingly, this suggests that rationing is quite important when the clearance price is too low in comparison to the regular price. Third, by considering only rationing levels 0 and 1, the revenue obtained is close to that of using an optimal rationing policy. In fact, v^{TE} coincides with v^{FP} for $0 < p_2 \leq 0.2$. In §6.1 we will see that similar observations hold when the firm has finite capacity.

4.2.   Exponential Demand Curve
An optimal rationing level for the exponential demand curve is given by Proposition 5.

PROPOSITION 5. *Suppose the demand curve is as specified in (13), and the prices are fixed at $p_1 > p_2 > 0$. Let*

$$\alpha_1 = \frac{p_2[D(p_2) - D(p_1)]}{1/e - D(p_1)p_2}$$

$$\alpha_2 = \frac{p_2[D(p_2) - D(p_1)]}{p_1 D(p_1)[p_1 - p_2]}.$$

An optimal rationing level is as follows:

$$\theta^* = \begin{cases} 0 & \text{if } p_1 \geq 1, \, p_1 D(p_1) \geq R \\ 1 & \text{if } p_1 \geq 1, \, p_1 D(p_1) < R \\ 1 & \text{if } p_1 < 1, \, \bar{\alpha} \leq \alpha_1 \\ 0 & \text{if } p_1 < 1, \, \bar{\alpha} \geq \alpha_2 \\ \tilde{\theta} & \text{if } p_1 < 1, \, \alpha_1 < \bar{\alpha} < \alpha_2, \end{cases} \quad (20)$$

where if $p_1 < 1$ and $\alpha_1 < \bar{\alpha} < \alpha_2$, the rationing level $\tilde{\theta} \in (0, \hat{\theta})$ is the solution to

$$p_2[D(p_2) - \alpha D(p_1)] - \bar{\alpha} D(r(\theta))\left[p_2 + \frac{(p_1 - p_2)r(\theta)}{1 - \theta}\right] = 0.$$

The corresponding revenue is

$$v^{\text{FP}} = \begin{cases} p_1 D(p_1) & \text{if } p_1 \geq 1, \, p_1 D(p_1) \geq R \\ R & \text{if } p_1 \geq 1, \, p_1 D(p_1) < R \\ R & \text{if } p_1 < 1, \, \bar{\alpha} \leq \alpha_1 \\ p_1 D(p_1) & \text{if } p_1 < 1, \, \bar{\alpha} \geq \alpha_2 \\ R + \bar{\alpha}(p_1 - p_2)D(r(\tilde{\theta}))[1 - r(\tilde{\theta})] \\ \qquad \text{if } p_1 < 1, \, \alpha_1 < \bar{\alpha} < \alpha_2. \end{cases} \quad (21)$$

Proposition 5 shows that when $p_1 < 1$, the optimal rationing level is 0 if the proportion of strategic customers is above a critical value α_2 and the optimal rationing level is 1 if the proportion of strategic customers is below a critical value α_1. When the proportion of strategic customers is in an intermediate range, the rationing level is nontrivial. This says that when many customers are strategic, the firm does not offer the product at the low price to deter them from switching, whereas when few customers are strategic, the firm gains from offering the product at low price to attract low-valuation customers.

5.   Finite Capacity with Pricing Flexibility
In this section we consider the setting where there is finite capacity $c < \infty$ and the seller determines the prices and rationing level. Having capacity c means that, at most, c units of the product can be sold. It will be convenient to model the rationing decision as choosing a proportion $\beta \in [0, 1]$ of remaining

capacity to make available in the clearance period. It can be shown that this is equivalent to choosing a sell-up-to limit for the clearance period.

As before, strategic customers time their purchases to maximize expected surplus. Such customers base their purchase decisions on their (common) perceived probability of obtaining the product at price p_2. Given a perceived probability $\theta \in [0, 1]$ of obtaining the product at price p_2, the demand at price p_1 is $d_1(\theta, p_1, p_2)$ and the demand at price p_2 is $d_2(\theta, p_1, p_2)$, where $d_1(\theta, p_1, p_2)$ and $d_2(\theta, p_1, p_2)$ are defined in (6) and (7). Note that θ is defined slightly differently here than in earlier sections.

The probability of obtaining the product at p_1 is irrelevant. A customer can always observe if the product is available at price p_1. If the product is available, then he can make his decision based on the probability of obtaining the product at p_2; if the product is not available, the customer has no choice other than to wait. Therefore, sales at regular price p_1 are

$$\hat{s}_1(\theta, p_1, p_2) = \min\{d_1(\theta, p_1, p_2), c\},$$

and sales at clearance price p_2 are

$$\hat{s}_2(\theta, p_1, p_2, \beta) = \min\{\beta[c - d_1(\theta, p_1, p_2)]^+, d_2(\theta, p_1, p_2)\}.$$

For fixed values of β and θ, the actual fraction of demand at price p_2 that is satisfied is

$$h(\theta, p_1, p_2, \beta) = \frac{\hat{s}_2(\theta, p_1, p_2, \beta)}{d_2(\theta, p_1, p_2)}, \tag{22}$$

when $d_2(\theta, p_1, p_2) > 0$. When $d_2(\theta, p_1, p_2) = 0$, we define $h(\theta, p_1, p_2, \beta) = \theta$ for convenience. In equilibrium, the actual fraction of demand satisfied at price p_2 should be equal to the common belief of customers; hence for a given value of β, an equilibrium value of θ satisfies

$$\theta = h(\theta, p_1, p_2, \beta). \tag{23}$$

Constraints such as (23) have been used in several recent papers; see, e.g., Liu and van Ryzin (2005), Zohar et al. (2002), Dana and Petruzzi (2001). In view of (23), we will continue to call θ the rationing level.

Denote the seller's revenue function by

$$\Gamma(\theta, p_1, p_2, \beta) = p_1\hat{s}_1(\theta, p_1, p_2) + p_2\hat{s}_2(\theta, p_1, p_2, \beta). \tag{24}$$

Supposing the seller is limited to deterministic choice of β, the optimization problem is

$$z = \max_{\theta \in [0,1], \beta \in [0,1], 0 \leq p_2 \leq p_1} \{\Gamma(\theta, p_1, p_2, \beta):$$

$$\theta = h(\theta, p_1, p_2, \beta)\}. \tag{25}$$

Formulation (25) implies that when there are multiple solutions to (23) for given (p_1, p_2, β), the one that

yields the highest revenue will prevail. The issue is moot when there is a unique value $\theta = \theta(p_1, p_2, \beta)$ determined by (23) for each (p_1, p_2, β). In §6.1, we will show that this is the case for the linear demand curve.

Let $(p_1^\dagger, p_2^\dagger) \in \arg\max_{p_1 \geq p_2 \geq 0} R(p_1, p_2)$ be an optimal price pair ignoring capacity constraints. We have the following result based on solutions to the infinite capacity case.

PROPOSITION 6. *If* $c \geq D(p_2^\dagger)$, *then an optimal solution to* (25) *is* $(\theta, p_1, p_2, \beta) = (1, p_1^\dagger, p_2^\dagger, 1)$.

After some manipulations, we can eliminate β and (25) can be rewritten as in the following proposition.

PROPOSITION 7. *Suppose* $D(\cdot)$ *is continuous. Then*

$$z = \max_{\theta \in [0,1], 0 \leq p_1 \leq p_2} \{\Psi(\theta, p_1, p_2): \theta D(p_2)$$

$$+ (1 - \theta)d_1(\theta, p_1, p_2) \leq c, D(p_1) \leq c\}, \tag{26}$$

where z *is defined in* (25).

Explicit solution of (26) appears to be difficult because of the constraint

$$\theta D(p_2) + (1 - \theta)d_1(\theta, p_1, p_2) \leq c. \tag{27}$$

Next, we derive an upper bound on z by dualizing (27). The bound will subsequently be shown to be tight for some examples. For $\lambda \geq 0$, let $\Psi^\lambda(\theta, p_1, p_2) = \Psi(\theta, p_1, p_2) + \lambda[c - \theta D(p_2) - (1 - \theta)d_1(\theta, p_1, p_2)]$ and consider

$$z^\lambda = \max_{\theta \in [0,1],\, 0 \leq p_2 \leq p_1} \{\Psi^\lambda(\theta, p_1, p_2): D(p_1) \leq c\}. \tag{28}$$

It follows that $z^\lambda \geq z$ for $\lambda \geq 0$. Therefore, $z \leq \bar{z}$, where

$$\bar{z} = \min_{\lambda \geq 0} z^\lambda. \tag{29}$$

Let $\tilde{p}_1 = p_1 - \lambda$ and $\tilde{p}_2 = p_2 - \lambda$ and define $\tilde{D}(\cdot)$ to be the demand function such that $\tilde{D}(p - \lambda) = D(p)$. After rearranging terms, the objective in (28) can be written as

$$\Psi^\lambda(\theta, p_1, p_2) = \alpha\tilde{p}_1\tilde{D}(\tilde{p}_1) + \tilde{p}_2[\tilde{D}(\tilde{p}_2) - \alpha\tilde{D}(\tilde{p}_1)]$$

$$+ (1 - \theta)[\bar{\alpha}r(\theta, \tilde{p}_1, \tilde{p}_2)\tilde{D}(r(\theta, \tilde{p}_1, \tilde{p}_2))$$

$$- \tilde{p}_2[\tilde{D}(\tilde{p}_2) - \alpha\tilde{D}(\tilde{p}_1)]] + \lambda c. \tag{30}$$

Note that (30) resembles the objective function in the infinite capacity case (8)–(11), except for the λc term; (30) can be interpreted as the total revenue when each unit of capacity earns revenue λ whether it is used or not and the price for each customer is reduced by λ. Let

$$\tilde{R}^\lambda(p_1, p_2) = \alpha p_1\tilde{D}(p_1) + p_2[\tilde{D}(p_2) - \alpha\tilde{D}(p_1)] + \lambda c$$

$$R^\lambda(p_1, p_2) = \alpha(p_1 - \lambda)D(p_1) + (p_2 - \lambda)$$

$$\cdot [D(p_2) - \alpha D(p_1)] + \lambda c.$$

A proof similar to that of Proposition 1 shows that

$$z^\lambda = \max_{p_1 \geq p_2 \geq -\lambda} \{\widetilde{R}^\lambda(p_1, p_2) : \widetilde{D}(p_1) \leq c\}$$

$$= \max_{p_1 \geq p_2 \geq 0} \{R^\lambda(p_1, p_2) : D(p_1) \leq c\}. \quad (31)$$

Equation (31) implies that a rationing level of $\theta = 1$ is optimal in (28) for each given $\lambda \geq 0$. Hence, $\theta = 1$ is optimal for the λ that minimizes (29). When the bound $\bar{z}$ is tight (i.e., $\bar{z} = z$), it follows that $\theta = 1$ is optimal in (25), in which case there is no added benefit to rationing (as we saw for the infinite capacity case in Proposition 1). Below, we show that $\bar{z}$ is indeed tight for the linear and exponential demand curves. However, there are demand curves for which (in contrast to the infinite capacity setting) rationing provides a strictly positive benefit when there is pricing flexibility. This is the case, for instance, when $\alpha = 0$, $c = 0.44$, and $D(p) = 0.8 - 2p^2$ for $p \in [0, \gamma]$, $D(p) = 0.75(1 - p)$ for $p \in (\gamma, 1]$, and $D(p) = 0$ otherwise, where $\gamma = (15 + \sqrt{385})/80 \approx 0.4328$. To see this, note that when $\alpha = 0$ the best solution with $\theta = 1$ in (26)—or (25)—yields revenue

$$\max_{p \geq 0} \{pD(p) : D(p) \leq c\} = \max_{p \geq 0} \{pD(p) : p \geq \rho\}, \quad (32)$$

where $\rho = D^{-1}(c) \approx 0.4243$. Note that $pD(p)$ has local maxima at $p = 2/\sqrt{30}$ and $p = 1/2$, which give respective objective values of $32/(30)^{3/2} \approx 0.1947$ and $3/16 = 0.1875$. The first of these is infeasible in (32), so the optimal solution to (32) is the larger of $\rho D(\rho) \approx 0.1867$ and 0.1875, which is 0.1875. Hence, the best solution in (25) with $\theta = 1$ yields revenue 0.1875. Direct calculations show that $(\theta, p_1, p_2) = (0.3, 0.5, 0.4)$ yields a better objective value of $\Psi(0.3, 0.5, 0.4) \approx 0.1879$ in (26) and is feasible in (25) with $\beta \approx 0.4235$ so that $z \geq \Gamma(0.3, 0.4, 0.5, 0.4235) \approx 0.1879$. Hence, there is strictly positive benefit to having a rationing level other than 1 in (25).

5.1. Linear Demand Curve
For the linear demand curve, we have $R^\lambda(p_1, p_2) = \alpha(p_1 - p_2)(1 - p_1) + (p_2 - \lambda)(1 - p_2) + \lambda c$ for $1 \geq p_1 \geq p_2 \geq 0$. From Proposition 6 and the analysis in §3.1, it follows that $(1, (3 - \alpha)/(4 - \alpha), (2 - \alpha)/(4 - \alpha), 1)$ is an optimal solution to (26) when $c \geq 2/(4 - \alpha)$. Hence, it remains to consider the case $c \leq 2/(4 - \alpha)$. Ignoring the constraints in (31) and assuming the expression for $R^\lambda(p_1, p_2)$ is valid for all p_1 and p_2, an optimal solution is

$$(p_1^\lambda, p_2^\lambda) = \left(\frac{3 - \alpha + \lambda}{4 - \alpha}, \frac{2 - \alpha - 2\lambda}{4 - \alpha}\right),$$

$$R^\lambda(p_1^\lambda, p_2^\lambda) = \frac{(1 - \lambda)^2}{4 - \alpha} + \lambda c.$$

Minimizing $R^\lambda(p_1^\lambda, p_2^\lambda)$ over all λ yields $\lambda^* = 1 - (4 - \alpha)c/2$. It can be checked that when $c \leq 2/(4 - \alpha)$, the constraints in (31) with $\lambda = \lambda^*$ are satisfied by

$$(p_1^*, p_2^*) = (p_1^{\lambda^*}, p_2^{\lambda^*}) = \left(1 - \frac{c}{2}, 1 - c\right). \quad (33)$$

Furthermore, $(1, p_1^*, p_2^*)$ also satisfies $p_1^* \leq 1$ as well as the constraints in (26) with the constraint (27) tight. Therefore, $(1, p_1^*, p_2^*)$ is an optimal solution to (26), and $z = \bar{z} = R^{\lambda^*}(p_1^{\lambda^*}, p_2^{\lambda^*}) = c - (4 - \alpha)c^2/4$.

5.2. Exponential Demand Curve
For exponential demand, the solution can easily be obtained, as in the previous subsection, using Proposition 6 and the analysis of §3.2 when $c \geq e^{\alpha/e-1}$. Suppose that $c \leq e^{\alpha/e-1}$; then an optimal solution to (31) is

$$(p_1^\lambda, p_2^\lambda) = \left(2 - \frac{\alpha}{e} + \lambda, 1 - \frac{\alpha}{e} + \lambda\right),$$

$$R^\lambda(p_1^\lambda, p_2^\lambda) = e^{-2+\alpha/e-\lambda} + \lambda c.$$

An optimal solution to (29) is $\lambda^* = -1 - \ln c + \alpha e^{-1}$ and $\bar{z} = (\alpha e^{-1} - \ln c)c$. Optimal prices in (26) are given by $(p_1^*, p_2^*) = (p_1^{\lambda^*}, p_2^{\lambda^*}) = (1 - \ln c, -\ln c)$ and $z = \bar{z}$.

6. Finite Capacity with Fixed Prices
We assume in this section that capacity is $c < \infty$ and that prices $p_1 > p_2 > 0$ are fixed. Let $\Theta = \{\theta : \theta = h(\theta, \beta)$ for some $\beta \in [0, 1]\}$ be the set of attainable values of θ. Hereafter, we consider cases where $\Theta = [0, \bar{\theta}]$ for some $\bar{\theta} \in [0, 1]$. This will be shown to be the case for the linear demand curve. Using (23) and the definition of h in (22), we can again eliminate β and state the optimization problem as

$$w^{\mathrm{FP}} = \max_{\theta \in [0, \bar{\theta}]} \Pi(\theta), \quad (34)$$

where

$$\Pi(\theta) = p_1 \hat{s}_1(\theta) + p_2 \theta d_2(\theta). \quad (35)$$

Observe that the seller's action β does not appear in (34). However, an optimal β can be determined in terms of an optimal solution θ^* in (34).

Note that if $c \leq D(p_1)$, then $\Pi(\theta) \leq p_1 c = \Pi(0)$, in which case an optimal solution to (34) is trivially $\theta = 0$. On the other hand, if $c > D(p_1)$, then $\hat{s}_1(\theta) = d_1(\theta)$, and therefore $\Pi(\theta) = \Psi(\theta)$. If also $D(p_1) < c < D(p_2)$, then $0 < \bar{\theta} < 1$. Finally, if $D(p_2) \leq c$ then $\bar{\theta} = 1$. Consequently, (34) is the same as (14); that is, $w^{\mathrm{FP}} = v^{\mathrm{FP}}$. Summarizing, we have

$$w^{\mathrm{FP}} = \begin{cases} p_1 c & \text{if } D(p_1) \geq c \\ \max\{\Psi(\theta) : 0 \leq \theta \leq \bar{\theta}\} & \text{if } D(p_1) < c < D(p_2) \\ v^{\mathrm{FP}} & \text{if } D(p_2) \leq c. \end{cases} \quad (36)$$

The optimization problem is very similar to the case with infinite capacity, except possibly a constraint on the range of θ values. Hence, for certain instances of the problem, we can solve the finite capacity case with the help of the solution to the infinite capacity case. Hereafter, we assume $D(p_1) < c$, because the problem is trivial when $D(p_1) \geq c$; see (36).

6.1. Linear Demand Curve

We begin with two lemmas. All proofs for this section are in the Online Supplement available at http://www.poms.org/journal/supplements.

LEMMA 1. *Suppose $D(p)$ is linear as in (12) and $0 < p_2 < p_1 < 1$ with $1 - p_1 < c$. Then*
1. *$h(\cdot, \beta)$ has a unique fixed point, and*
2. *the set Θ is an interval $[0, \bar{\theta}]$ for some $\bar{\theta} \in [0, 1]$. If also $c \geq 1 - p_2$, then $\Theta = [0, 1]$.*

LEMMA 2. *Suppose $D(p)$ is linear as in (12) and $0 < p_2 < p_1 < 1$ with $1 - p_1 < c$. Let*

$$c_1 = 1 - p_1 + \frac{\alpha(1 - p_1)(p_1 - p_2)}{1 - p_2}.$$

Then

$$\bar{\theta} = \begin{cases} \dfrac{c + p_1 - 1}{\alpha(p_1 - p_2)} & \text{if } 1 - p_1 < c \leq c_1, 0 < \alpha \leq 1 \\[2mm] \dfrac{c + \alpha p_1 - \alpha}{\alpha p_1 - p_2 + 1 - \alpha} & \text{if } c_1 < c < 1 - p_2, 0 \leq \alpha \leq 1 \\[2mm] 1 & \text{if } c \geq 1 - p_2. \end{cases} \quad (37)$$

Furthermore, $\bar{\theta} \leq \hat{\theta}$ if $c \leq c_1$, and $\bar{\theta} > \hat{\theta}$ if $c > c_1$.

When $D(p_1) \geq c$, an optimal rationing level is 0. When $D(p_2) \leq c$, the solution is given by Proposition 4; see also (36). Therefore, we only need to take care of the case where $D(p_1) < c < D(p_2)$. Proposition 8 gives the optimal rationing level in this case.

PROPOSITION 8. *Suppose the demand curve is linear and $1 - p_1 < c < 1 - p_2$. Let $\tilde{\theta}$ be as in (46) in the appendix, θ^* as in (19), and $\bar{\theta}$ as in (37).*
(i) *If $\alpha = 1$, then an optimal rationing level is $\bar{\theta}$.*
(ii) *If $\alpha = 0$, then an optimal rationing level is given by*

$$\theta^{**} = \begin{cases} 0 & \text{if } p_1 + p_2 < 1 \\ 0 & \text{if } p_1 + p_2 \geq 1, c < c_2 \\ \bar{\theta} & \text{if } p_1 + p_2 \geq 1, c \leq c_2, \end{cases} \quad (38)$$

where

$$c_2 = 1 - p_2 + \frac{(p_1 - p_2)(1 - p_1)}{p_2}.$$

(iii) *If $0 < \alpha < 1$, then an optimal rationing level is given by*

$$\theta^{**} = \begin{cases} 0 & \text{if } c \leq 1 - p_1 \\ \tilde{\theta} & \text{if } \tilde{\theta} \leq \bar{\theta}, 1 - p_1 < c \leq c_1 \\ \bar{\theta} & \text{if } \tilde{\theta} > \bar{\theta}, 1 - p_1 < c \leq c_1 \\ \tilde{\theta} & \text{if } \Psi(\tilde{\theta}) \geq \Psi(\bar{\theta}), c_1 < c < 1 - p_1 \\ \bar{\theta} & \text{if } \Psi(\tilde{\theta}) < \Psi(\bar{\theta}), c_1 < c < 1 - p_1 \\ \theta^* & \text{if } c \geq 1 - p_2. \end{cases} \quad (39)$$

In several of the cases in (39), θ^{**} is either 0 or the maximum value possible (1 or $\bar{\theta}$). This suggests the heuristic policy that selects θ from the better of the two extreme values.

6.1.1. The Two-Extreme Policy. Motivated by the previous discussion, a simple heuristic for the optimization problem in (34) is to either use $\theta = 0$ or $\theta = \bar{\theta}$. Let w^{TE} be the associated revenue; i.e.,

$$w^{\text{TE}} = \max\{\Pi(0), \Pi(\bar{\theta})\}. \quad (40)$$

We call the policy determined by (40) the *two-extreme policy*. Note that for a given price pair (p_1, p_2), if the capacity is not constraining, then $\bar{\theta} = 1$. Once $\bar{\theta}$ is determined, the optimization problem in (40) is easy because it requires computing $\Pi(\theta)$ for just two values of θ. Moreover, such a policy is easy to implement. To achieve $\theta = 0$, the firm's action is $\beta = 0$; that is, it does not sell in the clearance period. To achieve $\theta = 1$, the firm's action is $\beta = 1$; that is, it sells to any customers that seek to buy in the clearance period.

To test the effectiveness of two-extreme policies, we conducted numerical experiments using both linear and exponential demand functions. In the linear case, we solved the problem using the expressions in Proposition 8 and in the exponential case, we used a numerical search. For each case, we considered prices (p_1, p_2) such that $p_2 < p_1$ and $(p_1, p_2) \in \{0.05, 0.10, \ldots, 0.95\} \times \{0.05, 0.10, \ldots, 0.90\}$. We also considered $\alpha \in \{0.2, 0.5, 0.8\}$. The capacity was taken to be $c \in \{0.1, 0.2, \ldots, 1\}$. Therefore, for each fixed capacity level and α value, there are 171 different price pairs. Table 1 reports numerical results for the linear demand curve. For each fixed capacity level and α value, we report statistics on the relative difference $RD = 100 \times (w^{\text{FP}} - w^{\text{TE}})/w^{\text{FP}}$. In particular, we report the number of cases where the relative difference is 0, greater than 0.1, or greater than 1. We also list the maximum relative difference among all fixed price pairs for each capacity level and α value. In most cases, revenue from two-extreme policies is within 0.1% of the optimal revenue. The largest relative difference is 2.48%. The results are similar for

Table 1   **Relative Difference** $RD = 100 \times (w^{FP} - w^{TE})/w^{FP}$ **When Demand Curve Is Linear**

α	Capacity	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.2	# RD = 0	171	171	171	170	170	170	168	167	167	167
	# RD > 0.1	0	0	0	1	0	1	1	1	1	1
	Max RD	0.00	0.00	0.00	0.10	0.01	0.14	0.14	0.14	0.14	0.14
0.5	# RD = 0	171	168	167	166	164	165	164	160	158	158
	# RD > 0.1	0	3	4	5	7	6	7	11	13	13
	# RD > 1	0	1	1	2	3	2	2	2	2	2
	Max RD	0.00	1.41	1.20	2.20	2.48	1.51	1.14	1.14	1.14	1.14
0.8	# RD = 0	171	171	170	167	165	163	161	159	155	154
	# RD > 0.1	0	0	1	0	4	5	7	9	14	15
	# RD > 1	0	0	0	0	1	1	3	4	6	6
	Max RD	0.00	0.00	0.41	0.08	1.48	1.85	2.07	2.10	2.19	2.19

the exponential case, with the biggest relative difference at 2.28%. We do not include a table for the exponential case.

Overall, the results indicate that two-extreme policies yield revenue that is close to optimal. Given their simplicity to determine and implement, such two-extreme policies certainly appear to be fairly appealing. Note that in Table 1, the column with $c = 1$ corresponds to cases where capacity is nonconstraining, which were treated in §4.1. In such cases, a two-extreme policy becomes a 0-1 policy, where θ and β are either 0 or 1. In the notation of §4.1 the ($c = 1$) column shows $100 \times (v^{FP} - v^{TE})/v^{FP}$.

6.1.2.   The Cost of Ignoring Strategic Customer Behavior. If the firm ignores strategic behavior, then it will not hold back capacity from low-price customers; i.e., it will satisfy as many low-price customers as possible and $(100 \times \bar{\theta})\%$ of customers who seek to purchase at the low price will be satisfied. We call the policy that offers all capacity the *naive policy*. The revenue from a naive policy is $w^{NV} = \Gamma(\bar{\theta}, 1) = \Pi(\bar{\theta})$.

To investigate the performance of naive policies, we conducted a set of numerical experiments. The experimental setup is the same as that above. Table 2 reports the results when the demand curve is linear. (The exponential case is similar and again omitted.) We can see that naive policies are optimal in many cases, which should not be surprising in view of the results for the two-extreme policies. However, when naive policies are not optimal, the potential revenue loss from using such a policy can be as high as approximately 70%. This again shows that ignoring strategic customer behavior can have serious consequences. It also shows the importance of considering both extremes in the two-extreme policies described in the previous section. In the table, the instances for which the naive policy are particularly ineffective and for which the two-extreme policy—and rationing

Table 2   **Relative Difference** $RD = 100 \times (w^{FP} - w^{NV})/w^{FP}$ **When Demand Curve Is Linear**

α	Capacity	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.2	# RD = 0	163	150	137	127	120	125	107	103	98	97
	# RD > 0.1	8	21	34	44	51	56	64	68	73	74
	# RD > 1	8	21	34	43	51	55	62	66	71	72
	# RD > 10	7	19	30	38	43	49	49	53	58	59
	# RD > 40	4	9	15	19	23	25	27	23	23	23
	# RD > 70	1	1	2	3	3	1	0	0	0	0
	Max RD	70.53	73.33	73.33	72.75	71.72	70.10	68.08	66.06	64.04	63.03
0.5	# RD = 0	165	153	144	136	132	128	124	121	118	118
	# RD > 0.1	6	18	27	35	39	43	47	50	53	53
	# RD > 1	6	17	25	35	38	43	45	49	52	52
	# RD > 10	5	13	17	24	25	28	31	30	34	35
	# RD > 40	1	1	2	3	2	3	3	0	0	0
	Max RD	42.11	41.67	43.75	42.67	42.86	42.42	41.41	39.58	37.50	36.46
0.8	# RD = 0	168	165	161	156	153	149	147	146	144	144
	# RD > 0.1	3	6	10	12	17	20	22	23	27	27
	# RD > 1	3	6	8	12	14	17	19	20	23	24
	# RD > 10	1	1	2	2	3	3	4	4	6	5
	Max RD	13.68	13.33	12.94	14.67	13.81	14.17	14.00	13.33	12.00	10.71

in general—offers the most benefit are those in which p_2 is very small in relation to p_1. For example, for $c = 0.5$, $\alpha = 0.5$, $p_1 = 0.7$, and $p_2 = 0.1$, the revenue loss from the naive policy is about 33.33%; but for $p_2 \geq 0.3$, the naive policy is optimal. Note again that the $c = 1$ case reduces to the infinite capacity problem, so the rightmost column in the table shows $100 \times [v^{\mathrm{FP}} - \Psi(1)]/v^{\mathrm{FP}}$.

7. Summary

Customers may respond strategically to a seller's pricing and availability decisions. This suggests there is a need to incorporate customer responses in the decision-making process. In this paper, we analyzed a deterministic clearance sale model for a firm selling a single product to a population of customers, a fraction of which is strategic. We considered several versions of the problem, including cases with and without pricing flexibility. Among the insights that emerged from our study are that (1) when optimizing over prices, rationing offers no additional benefit to the firm when the firm has ample capacity; (2) for fixed prices, restricting availability at the clearance price—even when there is leftover inventory that will otherwise be wasted—can be advantageous to the firm; (3) most—but generally not all—of the benefit from such rationing can be obtained by using policies that make either all or none of the remaining inventory available in the clearance period; and (4) rationing can compensate only somewhat for poor pricing decisions but is nevertheless particularly important if the clearance price is set too low in relation to the regular price.

Appendix. Proofs

PROOF OF PROPOSITION 1. Let $\hat{p}(r) = \arg\max_{p \geq r} pD(p)$ and define

$$\Phi(\theta, p_1, p_2) = R(p_1, p_2) + (1 - \theta)\widehat{\Omega}(p_1, p_2, \hat{p}(p_1)). \quad (41)$$

Note that $\Psi(\theta, p_1, p_2) \leq \Phi(\theta, p_1, p_2)$ for all $\theta \in [0, 1]$, $0 \leq p_1 \leq p_2$ by (11) and the fact that $\hat{p}(p_1)D(\hat{p}(p_1)) \geq qD(q)$ for all $q \geq p_1$. Hence,

$$V \leq \max_{\theta \in [0, 1], 0 \leq p_2 \leq p_1} \Phi(\theta, p_1, p_2). \quad (42)$$

Let (θ^*, p_1^*, p_2^*) be an optimal solution of the optimization on the right-hand side in (42) and define $\hat{p}^* = \hat{p}(p_1^*)$. It follows that

$$\Phi(\theta^*, p_1^*, p_2^*) = R(p_1^*, p_2^*) + (1 - \theta^*)\widehat{\Omega}(p_1^*, p_2^*, \hat{p}^*)$$

$$= \begin{cases} R(p_1^*, p_2^*) & \text{if } \widehat{\Omega}(p_1^*, p_2^*, \hat{p}^*) \leq 0 \\ \alpha p_1^* D(p_1^*) + \bar{\alpha}\hat{p}^* D(\hat{p}^*) & \text{if } \widehat{\Omega}(p_1^*, p_2^*, \hat{p}^*) > 0. \end{cases}$$

From (2) we have $R(p_1^*, p_2^*) \leq v^\dagger$. We also have

$$\alpha p_1^* D(p_1^*) + \bar{\alpha}\hat{p}^* D(\hat{p}^*) \leq \max\{p_1^* D(p_1^*), \hat{p}^* D(\hat{p}^*)\}$$

$$\leq \max_{p \geq 0} pD(p) \leq v^\dagger.$$

Therefore, we conclude that $V \leq \Phi(\theta^*, p_1^*, p_2^*) \leq v^\dagger$. Furthermore, it is easy to see that $V \geq v^\dagger$. Thus, $V = v^\dagger$. □

PROOF OF PROPOSITION 2. Using (9), it can be seen that $v^{\mathrm{FP}} > R$ if and only if there exists $\theta \in [0, 1)$ such that $\Omega(\theta) > 0$. For $\theta \in [0, 1)$, we have $r(\theta) = (p_1 - \theta p_2)/(1 - \theta) \in [p_1, \infty)$. Thus, alternatively, $v^{\mathrm{FP}} > R$ if and only if there exists $q \in [p_1, \infty)$ such that $\widehat{\Omega}(p_1, p_2, q) > 0$. □

PROOF OF PROPOSITION 3. Recall the definitions of $\Phi(\cdot)$ and $\hat{p}(\cdot)$ from the proof of Proposition 1. Let $\hat{p} = \hat{p}(p_1)$. Because $\Psi(\theta) \leq \Phi(\theta)$, we have

$$v^{\mathrm{FP}} \leq \max_{0 \leq \theta \leq 1} \Phi(\theta)$$

$$= \begin{cases} R & \text{if } \widehat{\Omega}(p_1, p_2, \hat{p}) \leq 0 \\ \alpha p_1 D(p_1) + \bar{\alpha}\hat{p}D(\hat{p}) & \text{if } \widehat{\Omega}(p_1, p_2, \hat{p}) > 0. \end{cases}$$

From (16), we have $\alpha p_1 D(p_1) + \bar{\alpha}\hat{p}D(\hat{p}) \leq p_1 D(p_1)$, completing the proof. □

PROOF OF PROPOSITION 4. If $\theta \in (\hat{\theta}, 1)$, then $r(\theta) > 1$, and therefore $(1 - \theta)\Omega(\theta) = -(1 - \theta)p_2[D(p_2) - \alpha D(p_1)] < 0$ by (10)–(11) and (12). We therefore have

$$v^{\mathrm{FP}} = \max_{0 \leq \theta \leq 1}\{R + (1 - \theta)\Omega(\theta)\}$$

$$= R + \max\left\{\max_{0 \leq \theta \leq \hat{\theta}}(1 - \theta)\Omega(\theta), \max_{\hat{\theta} < \theta < 1}(1 - \theta)\Omega(\theta), 0\right\}$$

$$= R + \max\left\{\max_{0 \leq \theta \leq \hat{\theta}}(1 - \theta)\Omega(\theta), 0\right\}$$

$$= R + \max\left\{\max_{p_1 \leq x \leq 1}\frac{p_1 - p_2}{x - p_2}[\bar{\alpha}xD(x) - p_2[D(p_2) - \alpha D(p_1)]], 0\right\}.$$

The final equality follows by a change of variable $x = (p_1 - \theta p_2)/(1 - \theta)$. If the solution of the outer maximization above is 0, then an optimal rationing level is 1. Otherwise, an optimal rationing level can be obtained by solving for θ in terms of an x that maximizes

$$F(x) = \frac{\bar{\alpha}xD(x) - p_2[D(p_2) - \alpha D(p_1)]}{x - p_2} \quad (43)$$

over $[p_1, 1]$. Consider the maximization problem

$$\max_{p_1 \leq x \leq 1} F(x). \quad (44)$$

Upon substituting (12) into (43), and going through some algebra, we obtain

$$F(x) = \bar{\alpha}(1 - x) - \frac{\alpha p_2(p_1 - p_2)}{x - p_2} - \bar{\alpha}p_2. \quad (45)$$

Therefore, $F(x)$ is concave on $[p_2, \infty)$. Furthermore, we have

$$\frac{dF(x)}{dx} = -\bar{\alpha} + \frac{\alpha p_2(p_1 - p_2)}{(x - p_2)^2}.$$

Case (i), $\alpha = 1$: Here all customers are myopic, and it is straightforward to show that $v^{\mathrm{FP}} = R$ and $\theta^* = 1$ is optimal.

Case (ii), $\alpha = 0$: Here all customers are strategic. We have

$$v^{\text{FP}} = (1 - p_2)p_2 + (p_1 - p_2)\max\left\{\max_{p_1 \leq x \leq 1} 1 - x - p_2, 0\right\}$$

$$= \begin{cases} p_2(1 - p_2) & \text{if } p_1 + p_2 \geq 1 \\ p_1(1 - p_1) & \text{if } p_1 + p_2 < 1, \end{cases}$$

and the expression for an optimal rationing level follows from the discussion preceding (43).

Case (iii), $0 < \alpha < 1$: By setting $dF(x)/dx = 0$, we see that $F(x)$ is maximized over $[p_2, \infty)$ at $\hat{x} = p_2 + z$, where

$$z = \sqrt{\frac{\alpha p_2(p_1 - p_2)}{\bar{\alpha}}}.$$

To incorporate the constraint $p_1 \leq x \leq 1$, we consider the following cases based on the fact that $F(x)$ is concave. Let $\tilde{x}$ denote the optimal solution of (44). First, if $z < p_1 - p_2$, then $\hat{x} < p_1$, it follows that $\tilde{x} = p_1$. If $p_1 - p_2 \leq z \leq 1 - p_2$, then $\hat{x} \in [p_1, 1]$ and hence $\tilde{x} = \hat{x}$. Finally, if $z > 1 - p_2$, then $\hat{x} > 1$; hence $\tilde{x} = 1$. The rationing level corresponding to $\tilde{x}$ is $\tilde{\theta} = (\tilde{x} - p_1)/(\tilde{x} - p_2)$. Therefore,

$$\tilde{\theta} = \begin{cases} 0 & \text{if } z < p_1 - p_2 \\ 1 - \sqrt{\dfrac{\bar{\alpha}(p_1 - p_2)}{\alpha p_2}} & \text{if } p_1 - p_2 \leq z \leq 1 - p_2 \\ \hat{\theta} & \text{otherwise.} \end{cases} \quad (46)$$

We obtain an optimal rationing level θ^* based on the expressions for $\tilde{\theta}$ in (46). We have $\theta^* = \tilde{\theta}$ if $F(\tilde{x}) \geq 0$ and $\theta^* = 1$ if $F(\tilde{x}) < 0$. We consider several different cases.

If $z < p_1 - p_2$, we have $F(\tilde{x}) = F(p_1)$ is nonnegative if and only if $p_1 D(p_1) \geq R$.

If $p_1 - p_2 \leq z \leq 1 - p_2$, $F(\tilde{x})$ is positive if and only if $\bar{\alpha}\tilde{x}D(\tilde{x}) - p_2[D(p_2) - \alpha D(p_1)] \geq 0$.
After some algebra, the condition reduces to

$$\sqrt{\alpha p_2(p_1 - p_2)}\left((1 - 2p_2)\sqrt{\bar{\alpha}} - 2\sqrt{\alpha p_2(p_1 - p_2)}\right) \geq 0.$$

Hence, $\theta^* = \tilde{\theta}$ if $p_1 - p_2 \leq z \leq 1/2 - p_2$, and $\theta^* = 1$ if $1/2 - p_2 < z \leq 1 - p_2$.

If $z > 1 - p_2$, plugging $\tilde{x} = 1$ into (45), we can easily see that $F(\tilde{x}) < 0$. Summarizing the above, we obtain (19). Expression (18) follows from $v^{\text{FP}} = \Psi(\theta^*)$. $\square$

PROOF OF PROPOSITION 5. First consider the case $p_1 \geq 1$. Since $pD(p) = pe^{-p}$ is decreasing for $p \geq 1$, by Proposition 3 we have $v^{\text{FP}} = \max\{\Psi(0), \Psi(1)\}$. It follows that if $\Psi(0) \geq \Psi(1)$, then $\theta^* = 0$ is optimal; if $\Psi(0) < \Psi(1)$, then $\theta^* = 1$ is optimal. This yields the first two cases in (20) and (21).

In the remainder of the proof, we consider the case $p_1 < 1$. If $\Omega(\hat{\theta}) \leq 0$, then $\Omega(\theta) \leq \Omega(\hat{\theta}) \leq 0$ by the definition of Ω in (10) because $r(\hat{\theta}) = 1$ and because $qD(q)$ is maximized at $q = 1$ for exponential demand curve. By Proposition 2, the optimal revenue is R, and we can take $\theta^* = 1$. Note that

$$\Omega(\hat{\theta}) = \frac{\bar{\alpha}}{e} - p_2[D(p_2) - \alpha D(p_1)]$$

$$= \bar{\alpha}\left[\frac{1}{e} - p_2 D(p_1)\right] - p_2[D(p_2) - D(p_1)].$$

Consequently, $\Omega(\hat{\theta}) \leq 0$ is equivalent to

$$\bar{\alpha} \leq \frac{p_2[D(p_2) - D(p_1)]}{1/e - p_2 D(p_1)}.$$

This yields the third case in (20) and (21).

Next, suppose $\Omega(\hat{\theta}) > 0$. When demand is exponential as in (13), then $\Omega(\theta) = \widehat{\Omega}(p_1, p_2, r(\theta))$ is decreasing in $r(\theta)$ for $r(\theta) \geq 1$. Hence $(1 - \theta)\Omega(\theta) \leq (1 - \hat{\theta})\Omega(\hat{\theta})$ for $\theta \in [\hat{\theta}, 1]$. It follows that

$$v^{\text{FP}} = R + \max_{\theta \in [0, 1]}(1 - \theta)\Omega(\theta) = R + \max_{\theta \in [0, \hat{\theta}]}(1 - \theta)\Omega(\theta). \quad (47)$$

We will next show that $(1 - \theta)\Omega(\theta)$ is concave on $[0, \hat{\theta}]$. Observe that $qD(q) = qe^{-q}$ is increasing concave on $[0, 1]$. In addition, $r(\theta) \in [0, 1]$ for $\theta \in [0, \hat{\theta}]$, and is increasing convex in θ. Therefore, $r(\theta)D(r(\theta))$ is increasing concave in θ on $[0, \hat{\theta}]$, and so $\Omega(\theta)$ is increasing concave in θ on $[0, \hat{\theta}]$ by (11). Hence, $d^2[(1 - \theta)\Omega(\theta)]/d\theta^2 = -2d\Omega(\theta)/d\theta + (1 - \theta)d^2\Omega(\theta)/d\theta^2 \leq 0$, and so $(1 - \theta)\Omega(\theta)$ is concave on $[0, \hat{\theta}]$.

In view of the above, it is sufficient to find the optimal solution of (47) through first-order conditions. Using $dD(p)/dp = -D(p)$ we have

$$m(\theta) \equiv \frac{d}{d\theta}(1 - \theta)\Omega(\theta)$$

$$= -\bar{\alpha}D(r(\theta))\left[p_2 + \frac{(p_1 - p_2)r(\theta)}{1 - \theta}\right] + p_2[D(p_2) - \alpha D(p_1)].$$

It then follows that

$$m(0) = -\bar{\alpha}D(p_1)[p_2 + p_1(p_1 - p_2)] + p_2[D(p_2) - \alpha D(p_1)]$$

$$= p_2[D(p_2) - D(p_1)] - \bar{\alpha}p_1 D(p_1)(p_1 - p_2) \quad (48)$$

and

$$m(\hat{\theta}) = -\Omega(\hat{\theta}). \quad (49)$$

Hence, $\Omega(\hat{\theta}) > 0$ implies $m(\hat{\theta}) < 0$. It can be checked that $m(0) \geq m(\hat{\theta})$. If

$$m(0) > 0 > m(\hat{\theta}), \quad (50)$$

then the solution $\tilde{\theta} \in (0, \hat{\theta})$ of

$$m(\theta) = 0 \quad (51)$$

is an optimal rationing level. Note that (50) is equivalent to

$$\frac{p_2[D(p_2) - D(p_1)]}{1/e - D(p_1)p_2} < \bar{\alpha} < \frac{p_2[D(p_2) - D(p_1)]}{p_1 D(p_1)[p_1 - p_2]}.$$

This gives us the fifth case in (20). Equation (51) implies that

$$p_2[D(p_2) - \alpha D(p_1)] = \bar{\alpha}D(r(\tilde{\theta}))\left[p_2 + \frac{(p_1 - p_2)r(\tilde{\theta})}{1 - \tilde{\theta}}\right];$$

so

$$v^{\text{FP}} = R + (1 - \tilde{\theta})\Omega(\tilde{\theta})$$

$$= R + (1 - \tilde{\theta})\left[\bar{\alpha}D(r(\tilde{\theta}))r(\tilde{\theta}) - \bar{\alpha}D(r(\tilde{\theta}))\left[p_2 + \frac{(p_1 - p_2)r(\tilde{\theta})}{1 - \tilde{\theta}}\right]\right]$$

$$= R + \bar{\alpha}(p_1 - p_2)D(r(\tilde{\theta}))[1 - r(\tilde{\theta})],$$

which is the fifth expression in (21). Finally, if $\bar{\alpha} \geq p_2[D(p_2) - D(p_1)]/[p_1 D(p_1)(p_1 - p_2)]$, then $0 \geq m(0) \geq m(\hat{\theta})$, and thus $\theta^* = 0$ is an optimal rationing level. This gives us the fourth case in (20) and (21) and completes the proof. $\square$

PROOF OF PROPOSITION 6. Observe that $\Gamma(\theta, p_1, p_2, \beta) \leq \Psi(\theta, p_1, p_2) \leq \Psi(1, p_1^\dagger, p_2^\dagger)$ for all feasible $(\theta, p_1, p_2, \beta)$. Moreover, $c \geq D(p_2^\dagger)$ implies that $\Gamma(1, p_1^\dagger, p_2^\dagger, 1) = \Psi(1, p_1^\dagger, p_2^\dagger)$ and that $h(1, p_1^\dagger, p_2^\dagger, 1) = 1$. Hence, $(1, p_1^\dagger, p_2^\dagger, 1)$ is optimal for (25). $\square$

PROOF OF PROPOSITION 7. From Lemma 3 below, we can add an additional constraint $D(p_1) \leq c$ to (25) without loss of optimality. After doing so, we divide (25) into two cases. Let

$$z_1 = \max_{\theta \in [0,1], \beta \in [0,1], 0 \leq p_2 \leq p_1} \{\Gamma(\theta, p_1, p_2, \beta):$$

$$\theta = h(\theta, p_1, p_2, \beta), D(p_2) \leq c\} \quad (52)$$

and

$$z_2 = \max_{\theta \in [0,1], \beta \in [0,1], 0 \leq p_2 \leq p_1} \{\Gamma(\theta, p_1, p_2, \beta):$$

$$\theta = h(\theta, p_1, p_2, \beta), D(p_1) \leq c \leq D(p_2)\}. \quad (53)$$

We have $z = \max\{z_1, z_2\}$.

When $D(p_1) \leq c$, we can replace the objective function with $\Psi(\theta, p_1, p_2)$. Using Lemma 4 below, we can rewrite (52) as

$$z_1 = \max_{\theta \in [0,1], 0 \leq p_2 \leq p_1} \{\Psi(\theta, p_1, p_2): D(p_2) \leq c\}. \quad (54)$$

Now we simplify (53). After some manipulation, the constraints $\theta = h(\theta, p_1, p_2, \beta)$ and $0 \leq \beta \leq 1$ in (53) can be replaced by

$$\theta D(p_2) + (1 - \theta) d_1(\theta, p_1, p_2) \leq c. \quad (55)$$

Hence (53) can be rewritten as

$$z_2 = \max_{\theta \in [0,1], 0 \leq p_2 \leq p_1} \{\Psi(\theta, p_1, p_2): \theta D(p_2) + (1 - \theta)$$

$$\cdot d_1(\theta, p_1, p_2) \leq c, D(p_1) \leq c \leq D(p_2)\}. \quad (56)$$

Because (55) is satisfied when $D(p_2) \leq c$, we can combine (54) and (56) to get (26). $\square$

LEMMA 3. *Suppose $D(\cdot)$ is continuous, and $(\theta^*, p_1^*, p_2^*, \beta^*)$ is an optimal solution to (25). Then $D(p_1^*) \leq c$.*

PROOF. Suppose $D(p_1^*) > c$. Let $p_1^c > p_1^*$ be such that $D(p_1^c) = c$. Then $\Gamma(\theta^*, p_1^*, p_2^*, \beta^*) \leq p_1^* \min\{D(p_1^*), c\} = p_1^* c < p_1^c c = \Gamma(0, p_1^c, 0, 0)$. Furthermore, it can be checked that $(0, p_1^c, 0, 0)$ is a feasible solution. This is a contradiction and completes the proof. $\square$

LEMMA 4. *For each $\theta \in [0, 1]$ and $0 \leq p_2 \leq p_1$ such that $D(p_2) \leq c$, there exists $\beta \in [0, 1]$ such that $\theta = h(\theta, p_1, p_2, \beta)$ is satisfied in (52).*

PROOF. Fix θ, p_1, p_2. If $d_2(\theta, p_1, p_2) = 0$, then $\theta = h(\theta, p_1, p_2, \beta)$ is satisfied for any β. Next, consider the case $d_2(\theta, p_1, p_2) > 0$. Note that $c - d_1(\theta, p_1, p_2) \geq 0$ because $D(p_2) \leq c$ in (52). Hence, $\theta = h(\theta, p_1, p_2, \beta)$ can be written as

$$\theta = \min\left\{\beta\left[\frac{c - d_1(\theta, p_1, p_2)}{d_2(\theta, p_1, p_2)}\right], 1\right\}. \quad (57)$$

The right side of (57) is continuous in β and takes the values 0 and 1 when β equals 0 and 1, respectively. Therefore, there exists β such that (57) is satisfied. $\square$

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